THE ELLIPTIC-BI-PARABOLIC PLANAR TRANSFER FOR ARTIFICIAL SATELLITES

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ABSTRACT

The "elliptic-bi-parabolic transfer" orbit for artificial satellites is an extension of the bi-parabolic transfer that uses a Swing-By with a natural satellite of the main body to reduce the amount of fuel required by the maneuver. The objective is to find the minimum cost trajectory, in terms of fuel consumed, to transfer a spacecraft from a parking orbit around a planet to an orbit around a natural satellite of this planet (in a first version of this maneuver) or to a higher orbit around the planet (in a second version of this maneuver). The steps involved in this maneuver are: the application of an impulse to make the spacecraft to escape from the parking orbit around the planet using an elliptic transfer that crosses the orbit of the natural satellite; a Swing-By with the natural satellite with the periapsis distance controlled to make the spacecraft to reach a first parabolic transfer orbit; a zero cost impulse applied at infinity to change the orbit to a second parabolic transfer orbit that makes the spacecraft to return close to the natural satellite or around the planet (depending on the version considered) at an altitude equal to the radius of the final orbit desired; a third impulse to circularize the final orbit. The derivation of analytic equations that calculate the fuel saved in this maneuver when compared to the standard Hohmann transfer is made for both versions and it is used to generate numerical examples. Graphics are built to show in more details the potential savings given by this technique. After that, the idea of using a natural satellite in the maneuver is applied to the problem of making a spacecraft to escape from the main planet to the interplanetary space with maximum velocity at infinity. Numerical examples to leave Earth in a trip to all the planets of the Solar System and the to interstellar space are shown and the savings are quantified.

INTRODUCTION

R. H. Goddard (1919) was one of the first researchers to work on the problem of optimal transfers of a spacecraft between two points. He proposed optimal approximate solutions for the problem of sending a rocket to high altitudes with minimum fuel consumption. The problem of optimal transfers (in the sense of reducing the fuel consumption) between two Keplerian coplanar orbits has been under investigation for more than 40 years. In particular, many papers solve this problem for an impulsive thrust system with a fixed number of impulses. The literature presents many solutions for

particular cases, like the Hohmann (1925) and the bi-elliptic (Hoelker and Silber, 1959; Shternfeld, 1959) transfers between two circular orbits and their variants for ellipses in particular geometry.

The original Hohmann solution was obtained for a bi-impulsive transfer between two circular and coplanar orbits. It is the most used result in orbital maneuvers and it is applied here to compare the results generated by the technique suggested in the present paper. This important transfer has the following steps:

a) In the initial orbit a $\Delta V_0 = V_0 \left| \sqrt{\dfrac{2(r_f/r_0)}{(r_f/r_0)+1}} - 1 \right|$ (where r_0 (r_f) is the radius of the initial (final) orbit

and V_0 is the velocity of the spacecraft when in its initial orbit) is applied in the direction of the motion of the spacecraft. With this impulse the spacecraft is inserted into an elliptical orbit with periapsis r_0 and apoapsis r_f;

b) The second impulse is applied when the spacecraft is at the apoapsis. The magnitude is

$\Delta V_f = V_0 \left| 1 - \sqrt{\dfrac{2}{(r_f/r_0)+1}} \right| \sqrt{r_0/r_f}$ and it circularizes the orbit. This result is largely used nowadays,

as a first approximation of more complex models. Later, Hoelker and Silber (1959) (and others) showed that this transfer was not the best in all cases. A detailed study of those transfers can be found in Marec (1979). Next, the Hohmann transfer was generalized to the elliptic case (transfer between two coaxial elliptic orbits) by Marchal (1965). Smith (1959) shows results for some other special cases, like coaxial and quasi-coaxial elliptic orbits, circular-elliptic orbits, and two quasi-circular orbits.

The three-impulse concept is introduced in the literature by Shternfeld (1959) in Russia. He derived the bi-elliptic transfer (according to Edelbaum, 1967). This transfer was later independently derived by Hoelker and Silber (1959) and Edelbaum (1959). All those researches show that it is possible to find a bi-elliptical transfer between two circular orbits that has a ΔV lower than the one for the Hohmann transfer, when the ratio between the radius of the initial and the final orbits is greater than 11.93875. Following the idea of more than two impulses, there is also the paper by Broucke and Prado (1995) that uses three or four impulses passing through infinity. Two papers that document and summarize the knowledge about impulsive transfers are the ones written by Edelbaum (1967) and Gobetz and Doll (1969).

The present paper studies the problem of orbital maneuvers where a celestial body (a natural satellite of the main body considered) is used to decrease the ΔV (fuel consumed) required to complete the specified maneuver. It is called the "elliptic-bi-parabolic transfer" maneuver. A transfer from the Earth to the Moon and a transfer between two orbits around the Earth are used as examples, but the results are valid for any system of primaries.

THE ELLIPTIC-BI-PARABOLIC TRANSFER

The elliptic-bi-parabolic transfer is an extension of the bi-parabolic transfer, that is the limit case of the bi-elliptic transfer invented by Shternfeld (1959) and Hoelker and Silber (1959). In their original version, they show how to make a transfer between two circular and coplanar orbits in three

impulses. The sequence is: i) The first impulse is applied to send the spacecraft from its initial orbit to an elliptic orbit with large periapse distance (parabolic in the limit case); ii) The second impulse is applied at the apoapse of this first transfer orbit and it puts the spacecraft in a second elliptic (parabolic in the limit case) transfer orbit with periapse tangential with its final orbit; iii) The third impulse is applied in the periapse of this second transfer orbit and it completes the capture of the satellite in its final orbit. It is shown (Marec, 1979) that this transfer is more economical (in terms of ΔV) than the Hohmann transfer, in some cases.

The elliptic-bi-parabolic transfer takes advantage of an intermediate swing-by with the secondary body to reduce the amount of fuel required for the maneuver. It is useful to transfer a spacecraft from a low orbit around a central body to another body (a natural satellite) in orbit around this same central body or to transfer the spacecraft between two orbits around the central body. A good application is the transfer of a space vehicle from LEO (Low Earth Orbit) to the Moon. To develop the equations involved in this transfer it is assumed that: i) The initial LEO is circular with radius r_0; ii) The space vehicle is in a Keplerian orbit around the central body, except for the duration of the swing-by at the target body; iii) The swing-by at the target body can be modeled by the two-body scattering (Prado, 1993 and 1995); iv) The propulsion system is the usual impulsive system, able to delivery an instantaneous increment of velocity ΔV; v) The second body (the satellite) is in a circular orbit with radius r_B, coplanar with the initial orbit of the spacecraft; vi) The final orbit desired for the spacecraft is a circular orbit with radius r_f around the primary or the secondary body.

First version: transfer to a satellite body

With those hypotheses, the complete transfer (for the case where the final orbit is around the satellite body) follows the steps:

i) From the initial circular parking orbit an impulse is applied to send the spacecraft to an elliptic Hohmann transfer to the target body. This impulse is tangential to the initial orbit, and the magnitude is given by Equation (1). This magnitude is the same one that is applied in the Hohmann transfer, because the intermediate transfer orbits are the same in both cases.

$$\Delta V_1 = \sqrt{\frac{2\mu_C r_B}{r_0(r_0 + r_B)}} - \sqrt{\frac{\mu_C}{r_0}} \tag{1}$$

The time to apply this impulse is chosen such that the spacecraft reaches the apoapse of its transfer orbit at the same time that the target body is passing by that point, to have a near-collision encounter;

ii) In this point, the spacecraft makes a swing-by with the target body to transform its elliptic orbit O_1 around the central body to a parabolic orbit (P_1). In a typical planar swing-by, there are three independent free parameters that can be varied to achieve the purposes of the maneuver: V_∞ (the velocity of the spacecraft relative to the satellite body, when it is entering its sphere of influence); r_p (the distance during the moment of the closest approach); the approach angle ψ (the angle between the velocity of the spacecraft during the moment of the closest approach and the velocity of the planet). See Fig. 1 for more details. In this particular case, the values for V_∞ and ψ are not free, since

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it is decided to approach the target body from a Hohmann transfer (to achieve the minimum ΔV for the first impulse). What is left to choose is r_p, and it has to be chosen in such way that the orbit after the encounter is parabolic. From the condition of the orbit of the spacecraft before the encounter, the information available is:

$$V_i = \sqrt{2\mu_C \left(\frac{1}{r_B} - \frac{1}{r_0 + r_B}\right)} \qquad V_\infty = \sqrt{\frac{\mu_C}{r_B}} - V_i \qquad (2\text{-}3)$$

where, V_i is the velocity of the spacecraft relative to the central body before the encounter, V_∞ is the same velocity relative to the target body and μ_c is the gravitational parameter of the central body. Equation (3) is valid because the velocity of the spacecraft and the target body are aligned, at the near-collision point. From the condition for the desired orbit for the spacecraft after the encounter (it has to be parabolic) it is possible to say that:

$$V_o = \sqrt{\frac{2\mu_C}{r_B}} \qquad (4)$$

where V_o is the velocity of the spacecraft relative to the central body after the encounter (parabolic escape velocity). Using the rules to add two vectors the value for δ (the turn angle of the swing-by, see Fig. 1) is found to be:

$$Cos(2\delta) = \frac{V_o^2 - V_S^2 - V_\infty^2}{2V_S V_\infty} \qquad (5)$$

where V_S is the velocity of the satellite body with respect to the central body. Now, with the value of δ, the desired value of r_p is found, from the equation:

$$r_p = \frac{\mu_T}{V_\infty^2}\left(\frac{1}{sin\delta} - 1\right) \qquad (6)$$

where μ_T is the gravitational parameter of the target body. The approach of the spacecraft has to be calculated to obtain a close encounter with the natural satellite with this distance;

iii) Then, the same principle used in the bi-parabolic transfer is used here. Theoretically, it is necessary to wait until the spacecraft reaches the infinity to apply a zero impulse to transfer the spacecraft to a new parabolic orbit, that will meet the natural satellite with a periapse distance equals to r_f. This maneuver has a zero ΔV (called ΔV_i in Fig. 2), because it is performed at infinity, where the gravitational force from the central body is zero;

iv) The last step is the insertion of the spacecraft in an orbit around the target body. The same principle from the bi-parabolic transfer is used again.

The V_∞ (the velocity of the spacecraft relative to the target body when entering its sphere of influence) is calculated by Equation (7); then a conic around the target body with periapse at r_f is constructed and an impulse at the periapse of this conic is applied, opposite to the motion of the spacecraft, to reduce its velocity to the circular velocity at r_f. The magnitude of this impulse is given by Equation (8).

$$V_\infty = \sqrt{\frac{2\mu_C}{r_B}} - \sqrt{\frac{\mu_C}{r_B}} = \sqrt{\frac{\mu_C}{r_B}}\left(\sqrt{2}-1\right) \tag{7}$$

$$\Delta V_2 = \sqrt{V_\infty^2 + \frac{2\mu_T}{r_f}} - \sqrt{\frac{\mu_T}{r_f}} = \sqrt{\frac{\mu_C}{r_B}\left(\sqrt{2}-1\right)^2 + \frac{2\mu_T}{r_f}} - \sqrt{\frac{\mu_T}{r_f}} \tag{8}$$

All those equations can be combined to offer an expression for the savings in ΔV between the standard Hohmann transfer and the elliptic-bi-parabolic transfer. The expression is:

$$\Delta V_{SAV} = \left| \sqrt{\left(\sqrt{\frac{2\mu_C r_0}{r_B(r_B+r_0)}} - \sqrt{\frac{\mu_C}{r_B}}\right)^2 + \frac{2\mu_T}{r_f}} - \sqrt{\frac{\mu_C}{r_B}\left(\sqrt{2}-1\right)^2 + \frac{2\mu_T}{r_f}} \right| \tag{9}$$

It is important to note that the first impulse is the same for both maneuvers, so Equation (9) represents the difference between the velocity of the spacecraft when approaching the satellite body from an elliptic orbit that belongs to the Hohmann transfer (the first term) and the velocity of the spacecraft when approaching the satellite body from a parabolic orbit that belongs to the elliptic-bi-parabolic transfer (the second term).
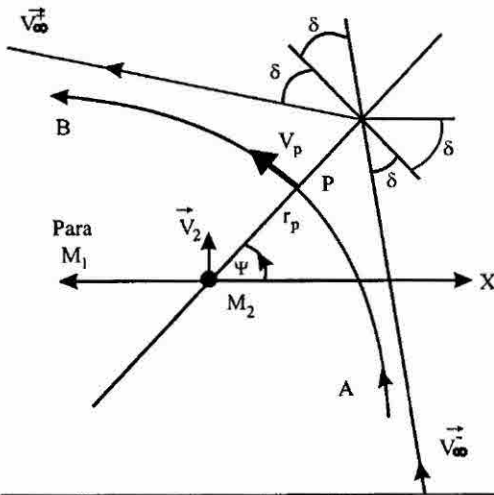

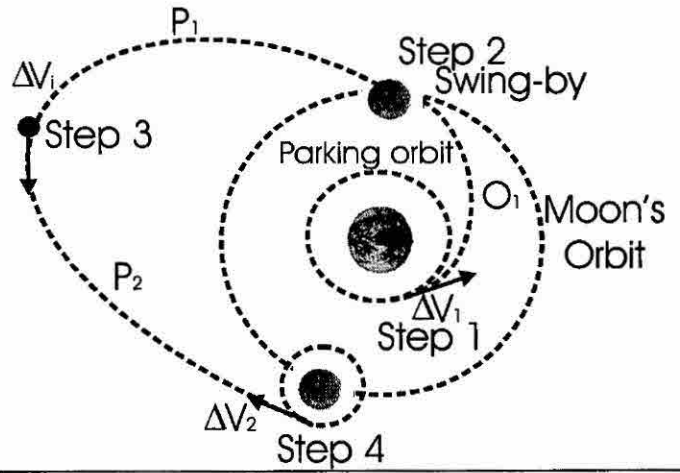| Fig. 1 – The geometry of the Swing-By.	Fig. 2 – Transfer from Earth to the Moon.

As an example, it is calculated the ΔVs involved to transfer a spacecraft from a circular low orbit around the Earth to a circular orbit around the Moon. The data used are: $r_0 = 6545$ km; $r_B = 384400$ km; $r_f = 1850$ km; $\mu_C = 398600.44$ km^3/s^2; $\mu_T = \mu_C/81.3$, where μ_C is the gravitational parameter of the

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main body and μ_T is the gravitational parameter of the natural satellite. The results are: ΔV_I = 3.140 km/s; V_i = 0.1863 km/s; V_∞ = 0.832 km/s; V_o = 1.440 km/s; δ = 39.13°; r_p = 4139.0 km; ΔV_2 = 0.713 km/s. The total ΔV involved in this maneuver is $\Delta V_T = \Delta V_I + \Delta V_2$ = 3.853 km/s. To give an idea of the savings obtained, Table 1 shows the standard results available, obtained from Sweetser (1991).

Table 1 - ΔV for several transfes in km/s

	ΔV_I	ΔV_2	ΔV_{total}
Hohmann	3.140	0.819	3.959
Bi-parabolic	3.232	0.714	3.946
Elliptic-Bi-Parabolic	3.140	0.713	3.853

To show better the possible savings in more generic cases, Fig. 3 shows contour-plots for the savings obtained. The canonical system of units is used in those graphs, what means that $\mu_C = r_0 = V_0 = 1$, where the unit for velocity is chosen to be V_0 (the velocity of a spacecraft in a circular orbit with radius r_0). The values of μ_T are 0.001, 0.01 and 0.1, respectively. The vertical axis is used for the variable r_f (the radius of the final circular orbit of the spacecraft around the natural satellite) and the horizontal axis is used for r_B (the radius of the circular orbit of the natural satellite around the planet).
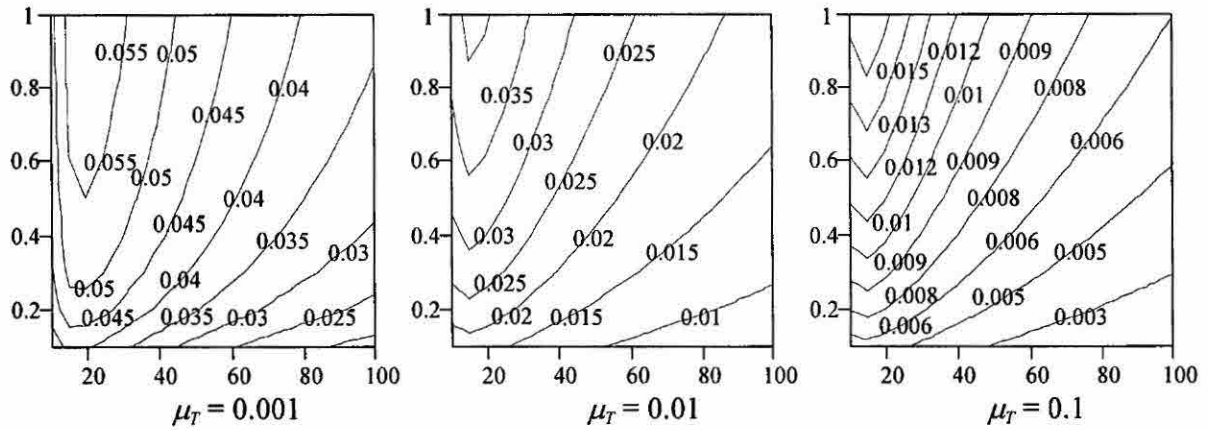

Fig. 3 - Contour-plots showing ΔV_{SAV} as a function of r_B (horizontal axis) and r_f (vertical axis).

Of course, this maneuver is not practical since the time required for the complete transfer is infinity. It should be considered as a limiting case of a more practical maneuver that performs the third step in a finite time (as large as possible) with $\Delta V \neq 0$ (but still very small). It is possible, in some cases, that the target body is not able to give the necessary impulse for the spacecraft to achieve parabolic orbit. In this case the swing-by maneuver can be used to get the maximum impulse possible to send the spacecraft to an elliptic orbit with the semi-major axis as large as possible, and the principles of the bi-elliptic transfer (Shternfeld, 1959 and Hoelker and Silber, 1959) are used to complete the maneuver. Another possible application for this transfer is a transfer between two planets, like an Earth-Mars transfer, using a swing-by in the target planet.

Second version: transfer between orbits around the primary body

Considering the case where the final orbit is around the primary body, the steps are (see Fig. 4):

i) From the initial circular parking orbit an impulse is applied to send the spacecraft to an elliptic Hohmann transfer to the target body. This step is the same one used in the first version, so the cost is given by Equation (1). The standard Hohmann transfer has a cost of $\Delta V_{H1} = \sqrt{\dfrac{2\mu_C r_f}{r_0(r_0 + r_f)}} - \sqrt{\dfrac{\mu_C}{r_0}}$.

ii) In this point, the spacecraft makes a swing-by with the target body to transform its elliptic orbit O_1 around the central body to a parabolic orbit (P_1), in the same way performed in the first version;

iii) Then, the same principle used in the bi-parabolic transfer is used again. Theoretically, it is necessary to wait until the spacecraft reaches the infinity to apply a zero impulse to transfer the spacecraft to a new parabolic orbit that will take the spacecraft at an altitude r_f. This maneuver has a zero ΔV (called ΔV_i in Fig. 4), because it is performed at infinity, where the gravitational force from the central body is zero;

iv) The last step is to circularize the orbit. For the Hohmann transfer the cost is $\Delta V_{H2} = \sqrt{\dfrac{2\mu_C r_0}{r_f(r_0 + r_f)}} - \sqrt{\dfrac{\mu_C}{r_f}}$ and for the elliptic-bi-parabolic transfer the cost is:

$$\Delta V_2 = \sqrt{\frac{2\mu_C}{r_f}} - \sqrt{\frac{\mu_C}{r_f}} = \sqrt{\frac{\mu_C}{r_f}}\left(\sqrt{2} - 1\right) \tag{10}$$

All those equations can be combined to offer an expression for the savings in ΔV between the standard Hohmann transfer and the elliptic-bi-parabolic transfer. The expression is given by Equation (11), where the first term represents the savings obtained in the second impulse and the second term represents the extra expenses of the first impulse.

$$\Delta V_{SAV} = \left(\sqrt{\frac{2\mu_C r_0}{r_f(r_f + r_0)}} - \sqrt{\frac{2\mu_C}{r_f}}\right) - \left(\sqrt{\frac{2\mu_C r_B}{r_0(r_B + r_0)}} - \sqrt{\frac{2\mu_C r_f}{r_0(r_f + r_0)}}\right) \tag{11}$$

To show better the possible savings in more generic cases, Fig. 5 shows contour plots for the savings obtained over the Hohmann transfer. The canonical system of units is used in those graphs, what means that $\mu_C = r_0 = V_0 = 1$, where the unit for velocity is chosen to be V_0 (the velocity of a spacecraft in a circular orbit with radius r_0). The vertical axis is used for the variable r_f (the radius of the final circular orbit of the spacecraft around the natural satellite) and the horizontal axis is used for r_B (the radius of the circular orbit of the natural satellite around the planet). Only situations where $r_B > r_f$ are shown in this plot. Positive numbers means a gain for the elliptic-bi-parabolic maneuver and a negative number means a gain for the Hohmann transfer.

As an example, it is calculated the ΔVs involved to transfer a spacecraft between two circular orbits around the Earth. The data used are: $r_0 = 6545$ km; $r_B = 384400$ km; r_f in the range 13090 km to 654500 km; $\mu_C = 398600.44$ km³/s², the gravitational parameter of the main body. The total ΔV involved in this maneuver is shown in Fig. 6, as a function of r_f, compared with the Hohmann and the bi-parabolic transfers. For the biparabolic transfer the impulses are given by:

$$\Delta V_1 = \sqrt{\frac{2\mu_C}{r_0}} - \sqrt{\frac{\mu_C}{r_0}} = \sqrt{\frac{\mu_C}{r_0}}\left(\sqrt{2}-1\right) \text{ and } \Delta V_2 = \sqrt{\frac{2\mu_C}{r_f}} - \sqrt{\frac{\mu_C}{r_f}} = \sqrt{\frac{\mu_C}{r_f}}\left(\sqrt{2}-1\right).$$ One canonical unit

of velocity corresponds to 7.8039 km/s. The results show that the elliptic-bi-parabolic transfer is better than the bi-parabolic in all situations, and the Hohmann transfer is the most economical only when $r_f < 10.37745$, that is a critical value for this maneuver. After this value the elliptic-bi-parabolic has the lowest cost, showing a difference about 0.5 canonical units (about 400 m/s) for $r_f > 40$.
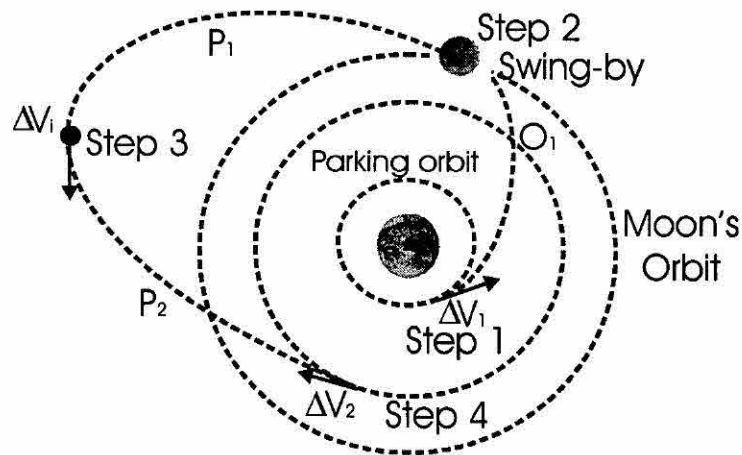

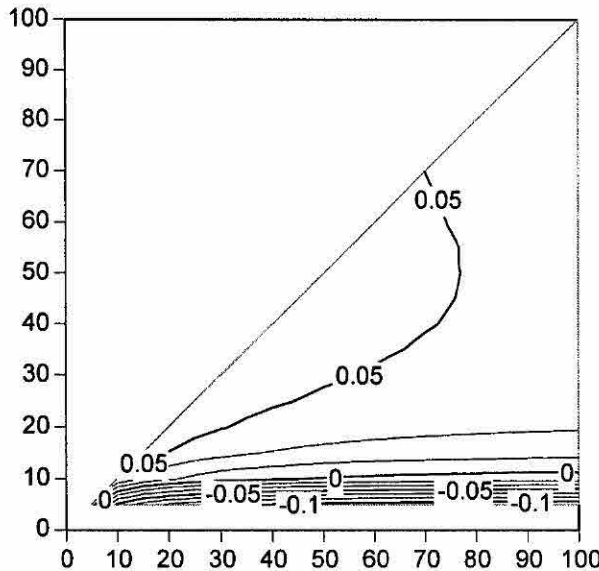Fig. 4 – Transfer between two orbits around the Earth.

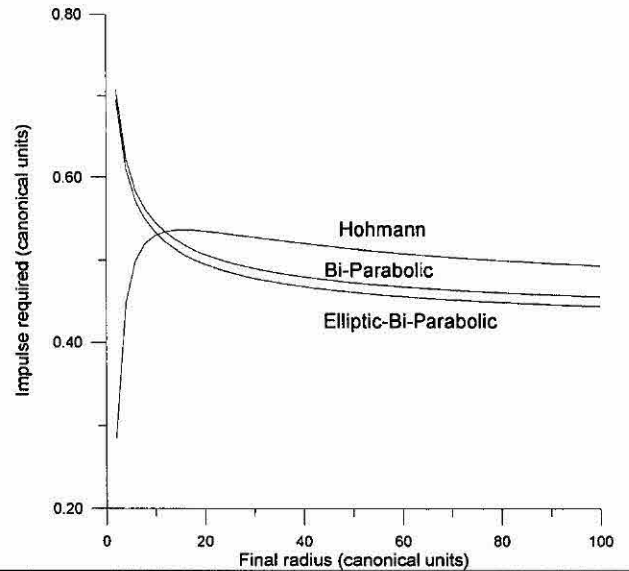| Fig.5-Savings obtained over the Hohmann Transfer. | Fig. 6 - ΔV involved in the example shown. |

The use of a swing-by to achieve escape velocity from the planet

A natural sequence of the maneuvers previously described is to use the swing-by with the satellite to achieve a hyperbolic orbit, instead of a parabolic one. In this way the spacecraft leaves the low circular orbit with an impulse a little bit smaller than the one required by the standard maneuver (with no swing-by), and it uses the natural satellite of the main body as an accelerator to compensate for this deficit. As an example, it is calculated the savings involved in a Hohmann transfer from Earth to all the planets in the Solar System and to the interstellar space, using the Moon as an accelerator. The standard procedure of the patched conic transfer is used and the results are shown in Table 2. Two values for the periapse distance during the swing-by (r_p) are used: 1750 km and 1840 km. They can show the importance of this parameter. The minimum possible value (about 1750 km, barely above the surface of the Moon) can provide better savings, but $r_p = 1840$ km can provide almost the same savings, with a comfortable distance greater than 100 km from the surface of the Moon during the swing-by (to avoid the risk of crashing into the Moon).

Table 2 - Savings in ΔV (m/s) in a Hohmann Transfer from Earth to the planets

Planet	ΔV for standard maneuver (km/s)	Savings (m/s, $r_p = 1750$ km)	Savings (m/s, $r_p = 1840$ km)
Mercury	5.561	53	52
Venus	3.511	142	137
Mars	3.619	129	124
Jupiter	6.310	44	43
Saturn	7.292	35	34
Uranus	7.984	31	30
Neptune	8.251	30	29
Pluto	8.367	29	28
Inter-stellar space	8.751	27	26

CONCLUSIONS

The use of the satellite of a planet to reduce the costs of several types of missions is explained. A maneuver that uses a swing-by with a natural satellite of a planet to transfer a spacecraft to the natural satellite or to a higher orbit around the planet was described. Analytical equations were derived and showed a savings in the order of 100 m/s in a transfer from Earth to the Moon and in the order of 400 m/s in transfers between two orbits around the Earth over the standart Hohmann transfer. It was shown that there is a critical value for r_f (10.37745) for the elliptic-bi-parabolic be more economical. After that, the idea of making a swing-by with the natural satellite was used in maneuvers to send a spacecraft to the other planets of the Solar System and to the interstellar space. The results show that the Moon is not a great accelerator, but savings in the order of 100 m/s can be achieved. Better results can be found for other hypothetical and real systems, where the natural satellite is a better accelerator.

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