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STUDY OF A PAIR OF DUAL MINIMAL REPRESENTATIONS FOR TRANSLATION INVARIANT SET MAPPINGS BY MATHEMATICAL MORPHOLOGY

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RESUMO - NOTAS / ABSTRACT - NOTES

In his 1975 book, Matheron introduced a pair of dual representation written in terms of elementary morphological mappings for increasing translation invariant (t.i.) set mappings using the concept of Kernel. Based on Hit-Miss topology, Maragos, in his 1985 Phd thesis, has given sufficient conditions on increasing t.i. mappings under which such mappings have minimal representations. In this report, a pair of dual representations written in terms of elementary morphological mappings for t.i. mappings (not necessarily increasing) is presented. It is shown that under the same sufficient conditions such mappings have minimal representations. Actually, the Matheron's and Maragos' representations are special cases of the proposed representations. Finally, some examples are given to illustrate the theory.

OBSERVAÇÕES/REMARKS

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RESUMO

No seu livro de 1975, Matheron introduziu um par de representações duais escritas em termos de mapeamentos morfológicos elementares para mapeamentos de conjuntos, invariantes em translação (i.t.) e crescentes usando o conceito de núcleo. Baseado na topologia Toca-Não Toca (Hit-Miss), Maragos, na sua tese de Phd de 1985, deu condições suficientes sobre mapeamentos i.t. crescentes que garantem que tais mapeamentos possuem representações minimais. Neste relatório, um par de representações duais escritas em termos de mapeamentos morfológicos elementares para mapeamentos i.t. (não necessariamente crescentes) é apresentado. Mostra-se que as mesmas condições suficientes garantem que tais mapeamentos possuem representações minimais. Na verdade, as representações de Matheron e Maragos são casos particulares das representações propostas. Finalmente, alguns exemplos são dados para ilustrar a teoria.

CONTENTS

LIST OF FIGURES

Page

CHAPTER 1

INTRODUCTION

Let E be a d-dimensional Euclidean space (e.g., $\mathbb{R}^d$), $\mathcal{A}$ be a collection of subsets of E, that is, $\mathcal{A} \subset \mathcal{P}(E)$, and ψ be a mapping from $\mathcal{A}$ to $\mathcal{P}(E)$. In the field of image processing, that motivates this paper, d is 2, $\mathcal{A}$ represents the collection of shapes, objects or images of interest (the terminology varies from author to author) and ψ represents a particular shape transformation.

The objective of this paper is to present a pair of minimal representations in terms of elementary mappings of the mathematical morphology (erosion and dilation) for ψ in the general class of *translation invariant* (t.i.) mappings (i.e., $\psi(X_h) = (\psi(X))_h$, where X_h represents the translate of X by a vector h of E), in the same way as Maragos (1985, 1989) and Dougherty and Giardina (1986) have done for ψ in the restricted class of *increasing* t.i. mappings (i.e., $X_1 \subset X_2 \Rightarrow \psi(X_1) \subset \psi(X_2)$). In image processing this may be important because common transformations, such as edge extraction or shape recognition, are not increasing.

Actually, Maragos' minimal representations are minimal forms of Matheron's representations for increasing t.i. mappings. Matheron (1975) has shown that any increasing t.i. mapping ψ can be represented as the supremum of a family of elementary mappings of the same type called erosions or as the infimum of a family of elementary mappings of the same type called dilations. In representation for ψ by a supremum, the structuring elements, which caracterize the erosions, belong to a set collection called kernel of ψ. The powerful concept of kernel, introduced by Matheron, consists of associating to each t.i. mapping ψ a subcollection of $\mathcal{A}$, the kernel of ψ, denoted $\mathcal{K}(\psi)$ and given by

$$\mathcal{K}(\psi) = \left\{X \in \mathcal{A}:\ o \in \psi(X)\right\},$$

where o is the null vector of E. Hence, for any increasing t.i. mapping ψ, the Matheron's representation by a supremum leads to the expression

$$\psi(X) = \bigcup \left\{X \ominus \check{A}:\ A \in \mathcal{K}(\psi)\right\} \quad (X \in \mathcal{A}),$$

where $X \ominus \check{A}$ is the erosion of X by the structuring element A (see Chapter 2 for the definition of erosion).

The Matheron's representation by a supremum works for three reasons: first, the t.i. assumption on ψ implies that the mapping $\mathcal{K}(\cdot)$ is a lattice-isomorphism (i.e., $\mathcal{K}(\cdot)$ is bijective, that is, one-to-one and onto, and increasing two-sided [1], that is,

$$\psi_1(X) \subset \psi_2(X)\ (X \in \mathcal{A}) \Leftrightarrow \mathcal{K}(\psi_1) \subset \mathcal{K}(\psi_2)\ );$$

second, the increasing assumption implies that $\mathcal{K}(\psi)$ is a dual ideal of $(\mathcal{A}, \subset)$ (i.e., if $X \in \mathcal{K}(\psi)$ and $Y \in \mathcal{A}$, then $X \subset Y$ implies that $Y \in \mathcal{K}(\psi)$); third, the kernel of erosion by A is the collection of all subsets of E in $\mathcal{A}$ which contain A.

When ψ is not increasing the Matheron's representation by a supremum fails, because the above second reason does not apply any more.

In this paper, it is shown that, by choosing a slightly different class of elementary mappings, any t.i. mapping (not necessarily increasing) also has a representation in terms of a supremum. More precisely, the proposed representation by a supremum leads to the

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[1] *two-sided stands for the double implication* ($\Leftrightarrow$).

expression

$$\psi(X) = \bigcup \left\{ X \circledast (A, B) : (A, B) \in \mathfrak{R}(\psi) \right\} \quad (X \in \mathscr{A}),$$

where $X \circledast (A, B)$ is the result of the intersection of the erosion of X by A and the erosion of X^c by B^c, that is,

$$X \circledast (A, B) = (X \ominus \check{A}) \cap (X^c \ominus \check{B}^c),$$

and $\mathfrak{R}(\psi)$ is a set of extremity pairs of the closed intervals contained in the kernel of ψ (see Section 3.1 for the definition of a closed interval). Furthermore, as in the case of increasing mappings, a dual representation for t.i. mappings (not necessarily increasing) is derived, in terms of the infimum of a family of dual elementary mappings. More precisely, the dual representation leads to the expression

$$\psi(X) = \bigcap \left\{ X \circledcirc (A, B) : (A, B) \in \mathfrak{R}(\psi^*) \right\} \quad (X \in \mathscr{A}),$$

where $\cdot \circledcirc (A, B)$ and ψ^* are the dual mappings, respectively, of $\cdot \circledast (A, B)$ and ψ (see end of Chapter 2 for the definition of dual).

One of the reasons for the general representation by a supremum to work is that the kernel of the elementary mapping $\cdot \circledast (A, B)$ is the collection of all subsets of E in $\mathscr{A}$ which are in between A and B. Compared to the kernel of the erosion by A, this kernel is "limited above by B" which is the key idea to set up the general representation by a supremum.

In his theory of minimal elements, Maragos has shown that Matheron's representations can be simplified in the sense that, usually, an increasing t.i. mapping ψ can be represented as the supremum of a *smaller* family of erosions or as the infimum of a *smaller* family of dilations. In the case of a supremum, for exemple, this occurs because the kernel of the erosion is decreasing with respect to its structuring element (i.e., $A_1 \subset A_2 \Rightarrow \mathcal{K}(\cdot \ominus \check{A}_1) \supset \mathcal{K}(\cdot \ominus \check{A}_2)$). A smaller family of erosions is then obtained by looking for the minimal elements of the kernel of ψ. The collection $\mathcal{B}(\psi)$ of the minimal elements of the kernel of ψ is called, by Maragos, the basis of ψ (Dougherty's and Giardina's basis definition is slightly different). Under a semi-continuity condition on ψ, Maragos has proved that the basis $\mathcal{B}(\psi)$ can be used to derive a minimal representation for increasing t.i. mappings leading to the expression

$$\psi(X) = \bigcup \left\{X \ominus \check{A}:\ A \in \mathcal{B}(\psi)\right\} \quad (X \in \mathcal{A}).$$

In the same way, the proposed representations for t.i. mappings (not necessarily increasing) appear to be redundant and minimal representations can be derived. In the case of a representation by a supremum, this occurs because the kernel of the elementary mapping $\cdot \circledast (A, B)$ is increasing with respect to its pair of structuring elements $\mathfrak{x} = (A, B)$, under some defined partial order, denoted $\{$ (i.e., $\mathfrak{x}_1 \{ \mathfrak{x}_2 \Rightarrow \mathcal{K}(\cdot \circledast \mathfrak{x}_1) \subset \mathcal{K}(\cdot \circledast \mathfrak{x}_2)$; see Section 3.1 for the definition of $\{$). In this paper, the collection $\mathfrak{B}(\psi)$ of maximal elements of $\mathfrak{K}(\psi)$ is called *basis* of ψ and it is shown that, under the same semi-continuity condition on ψ, the basis $\mathfrak{B}(\psi)$ can be used to derive a minimal representation for t.i. mappings leading to the expression

$$\psi(X) = \bigcup \left\{X \circledast (A, B):\ (A, B) \in \mathfrak{B}(\psi)\right\} \quad (X \in \mathcal{A}).$$

As in Maragos, the semi-continuity is expressed in terms of the Hit-Miss topology.

In Chapter 2 some useful known definitions and properties of the kernel of a t.i. mapping are recalled. In Chapter 3 the pair of dual representations for t.i mappings is derived. In Chapter 4, a new class of so-called inf-separable mappings is introduced, the cases of increasing, decreasing and inf-separable t.i. mappings are studied and, in the former case, the Matheron's representation by a supremum is derived from the proposed one. Chapter 5 contains the definition of t.i. mapping basis and sufficient conditions under which t.i. mappings have minimal representations. Finally, in Chapter 6, some simple examples are given to illustrate the theory.

The material in this paper is original except the one in Chapter 2.

CHAPTER 2

TRANSLATION INVARIANT MAPPINGS

All the main results in this chapter can be found in Matheron (1975). They are presented here for the sake of completeness and because of their fundamental role in this paper.

Let $\mathscr{A}$ be a non empty collection of subsets of a non empty set E, that is, $\mathscr{A} \subset \mathscr{P}(E)$, $\Psi_{\mathscr{A}}$ be the the set of all mappings $\psi(\cdot)$ or, simply, ψ from $\mathscr{A}$ to $\mathscr{P}(E)$ and $<$ be the partial order for $\Psi_{\mathscr{A}}$ defined by

$$\psi_1 < \psi_2 \text{ iff } \psi_1(X) \subset \psi_2(X) \quad (X \in \mathscr{A}).$$

The poset $(\Psi_{\mathscr{A}}, <)$ is a complete lattice. If $\sqcap \left\{\psi_i : i \in I\right\}$ and $\sqcup \left\{\psi_i : i \in I\right\}$ denote, respectively, the infimum and supremum of the family $\left\{\psi_i : i \in I\right\}$ of mappings in $\Psi_{\mathscr{A}}$, then

$$(\sqcap \left\{\psi_i : i \in I\right\})(X) = \cap \left\{\psi_i(X) : i \in I\right\} \quad (X \in \mathscr{A})$$

and

$$(\sqcup \left\{\psi_i : i \in I\right\})(X) = \cup \left\{\psi_i(X) : i \in I\right\} \quad (X \in \mathscr{A}).$$

In this paper, an important subclass of $\Psi_{\mathscr{A}}$ is studied, when the set E is an Abelian group with a binary operation, denoted +, and a zero element, denoted o. Some preliminary definitions are first recalled.

Let $h \in E$ and $X \in \mathscr{P}(E)$, then the set X_h given by

$$X_h = \left\{u \in E:\ u = h + x \text{ and } x \in X\right\}$$

or, equivalently,

$$X_h = \left\{u \in E:\ u - h \in X\right\} \tag{2.1}$$

is called the *translate* of X by h. In particular, $X_o = X$.

For any $\mathscr{A} \subset \mathscr{P}(E)$ and $h \in E$, let $\mathscr{A}_h$ denote the collection of translates of the elements of $\mathscr{A}$ by h, that is,

$$\mathscr{A}_h = \left\{X \in \mathscr{P}(E):\ X_{-h} \in \mathscr{A}\right\}. \tag{2.2}$$

In particular, $\mathscr{A}_o = \mathscr{A}$.

For any $h \in E$, $(\mathscr{A}_h)_{-h} = \mathscr{A}$ and $\mathscr{A}' \subset \mathscr{A} \Leftrightarrow \mathscr{A}'_h \subset \mathscr{A}_h$. This implies that intersection and union commute with translation, that is,

$$\bigcap \mathscr{A}_h = (\bigcap \mathscr{A})_h \text{ and } \bigcup \mathscr{A}_h = (\bigcup \mathscr{A})_h. \tag{2.3}$$

The collection $\mathscr{A} \subset \mathscr{P}(E)$ is said to be *closed under translation* iff for any $h \in E$, $\mathscr{A}_h = \mathscr{A}$.

Let $\mathscr{A} \subset \mathscr{P}(E)$ be closed under translation. A mapping ψ from $\mathscr{A}$ to $\mathscr{P}(E)$ is said to be *translation invariant* (t.i.) iff

$$\psi(X_h) = (\psi(X))_h \quad (X \in \mathscr{A},\ h \in E).$$

Let $\Phi'_{\mathscr{A}}$ denote the set of all the t.i. mappings from $\mathscr{A}$ (closed under translation) to $\mathscr{P}(E)$ ($\Phi_{\mathscr{A}} \subset \Psi_{\mathscr{A}}$). From (2.3), the infimum and the supremum of any family of t.i. mappings are t.i. mappings. Therefore, the subposet $(\Phi_{\mathscr{A}}, <)$ is also a complete lattice.

Let $\mathcal{K}(\cdot)$ be the mapping from $\Phi_{\mathcal{A}}$ to $\mathcal{P}(\mathcal{A})$ defined by

$$\mathcal{K}(\psi) = \left\{X \in \mathcal{A}:\ o \in \psi(X)\right\}, \tag{2.4}$$

for any $\psi \in \Phi_{\mathcal{A}}$. $\mathcal{K}(\psi)$ is called, by Matheron, the *kernel* of ψ.

In what follows, it is proved that the mapping $\mathcal{K}(\cdot)$ is a lattice isomorphism (i.e., a lattice-morphism and a bijection). Let us recall first the following important property of the kernel of a t.i. mapping.

PROPERTY 2.1 - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation, ψ be a t.i. mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ and $\mathcal{K}(\psi)$ be its kernel, defined by* (2.4). *For any* $X \in \mathcal{A}$,

$$x \in \psi(X) \text{ iff } X \in (\mathcal{K}(\psi))_x. \qquad \square$$

PROOF: For any $X \in \mathcal{A}$,

from (2.1), $\quad x \in \psi(X) \Leftrightarrow o \in (\psi(X))_{-x}$,

by t.i. definition, $\quad \Leftrightarrow o \in \psi(X_{-x})$,

from (2.4), $\quad \Leftrightarrow X_{-x} \in \mathcal{K}(\psi)$,

from (2.2), $\quad \Leftrightarrow X \in (\mathcal{K}(\psi))_x$. $\square$

Let $\phi_{\bullet}$ be the mapping from $\mathcal{P}(\mathcal{A})$ to $\Psi_{\mathcal{A}}$ defined by

$$\phi_{\mathcal{C}}(X) = \left\{x \in E:\ X \in \mathcal{C}_x\right\} \quad (X \in \mathcal{A}), \tag{2.5}$$

for any $\mathcal{C} \in \mathcal{P}(\mathcal{A})$.

This way of constructing a mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ is useful in the study of the properties of the mapping $\mathcal{K}(\cdot)$.

PROPERTY 2.2 - *let* $\mathcal{A} \subset \mathcal{P}(E)$ *be closed under translation,* ψ *be a t.i. mapping from* $\mathcal{A}$ *to* $\mathcal{P}(E)$ *and* $\mathcal{K}(\psi)$ *be its kernel, defined by (2.4), then the mapping* $\phi_{\mathcal{K}(\psi)}$ *from* $\mathcal{A}$ *to* $\mathcal{P}(E)$, *defined by (2.5) (with* $\mathcal{B} = \mathcal{K}(\psi)$*), is* ψ*, that is,*

$$\phi_{\mathcal{K}(\psi)} = \psi,$$

or, equivalently,

$$\psi(X) = \left\{x \in E:\ X \in (\mathcal{K}(\psi))_x\right\} \quad (X \in \mathcal{A}). \qquad \square$$

PROOF: For any $X \in \mathcal{A}$,

from (2.5), $\phi_{\mathcal{K}(\psi)} = \left\{x \in E:\ X \in (\mathcal{K}(\psi))_x\right\}$,

by Property 2.1, $= \left\{x \in E:\ x \in \psi(X)\right\}$,

$= \psi(X)$. $\square$

LEMMA 2.1 - *Let* $\mathcal{A} \subset \mathcal{P}(E)$. *The mapping* $\mathcal{K}(\cdot)$ *from* $\Phi_{\mathcal{A}}$ *to* $\mathcal{P}(\mathcal{A})$, *defined by (2.4), is injective (one to one).* $\square$

PROOF: Property 2.2 is a sufficient condition for the mapping $\mathcal{K}(\cdot)$ to be injective (see Property 6.3 p. 14 in Dugundji (1966)). $\square$

The mappings $\phi_{\mathcal{B}}$ from $\mathcal{A}$ to $\mathcal{P}(E)$, defined by (2.5), have the following property.

PROPERTY 2.3 - *Let* $\mathcal{A} \subset \mathcal{P}(E)$ *be closed under translation and* $\mathcal{B} \subset \mathcal{A}$. *The maping* $\phi_{\mathcal{B}}$ *from* $\mathcal{A}$ *to* $\mathcal{P}(E)$, *defined by (2.5), is t.i., that is,* $\phi_{\mathcal{B}} \in \Phi_{\mathcal{A}}$, *and its kernel, defined by (2.4), is* $\mathcal{B}$, *that is,*

$$\mathcal{K}(\phi_{\mathcal{B}}) = \mathcal{B}. \qquad \square$$

PROOF: 1. For any $x \in E$ and $X \in \mathcal{A}$,

from (2.5), $$\phi_{\mathcal{E}}(X_x) = \left\{u \in E:\ X_x \in \mathcal{E}_u\right\},$$

from (2.2), $$= \left\{u \in E:\ X \in \mathcal{E}_{u-x}\right\},$$

from (2.5), $$= \left\{u \in E:\ u - x \in \phi_{\mathcal{E}}(X)\right\},$$

from (2.1), $$= (\phi_{\mathcal{E}}(X))_x,$$

that is, $\phi_{\mathcal{E}}$ is t.i..

2. From (2.4),

$$\mathcal{K}(\phi_{\mathcal{E}}) = \left\{X \in \mathcal{P}(E):\ o \in \phi_{\mathcal{E}}(X)\right\},$$

from (2.5), $$= \left\{X \in \mathcal{P}(E):\ o \in \left\{x \in E:\ X \in \mathcal{E}_x\right\}\right\},$$

$$= \left\{X \in \mathcal{P}(E):\ X \in \mathcal{E}\right\} = \mathcal{E}.$$ □

LEMMA 2.2 - *let* $\mathcal{A} \subset \mathcal{P}(E)$. *The mapping* $\mathcal{K}(\cdot)$ *from* $\Phi_{\mathcal{A}}$ *to* $\mathcal{P}(\mathcal{A})$, *defined by* (2.4), *is surjective (onto).* □

PROOF: Property 2.3 is a sufficient condition for the mapping $\mathcal{K}(\cdot)$ to be surjective (see Property 6.9 p. 14 in Dugundji (1966)). □

LEMMA 2.3 (Matheron (1975)) - *Let* $\mathcal{A} \subset \mathcal{P}(E)$ *be closed under translation. The mapping* $\mathcal{K}(\cdot)$ *from* $\Phi_{\mathcal{A}}$ *to* $\mathcal{P}(\mathcal{A})$, *defined by* (2.4), *is bijective.* □

PROOF: This is a consequence of Lemmas 2.1 and 2.2. □

The following lemma states another important property of $\mathcal{K}(\cdot)$.

LEMMA 2.4 - *Let $\mathscr{A} \subset \mathscr{P}(E)$. the mapping $\mathscr{K}(\cdot)$ from $\Phi_{\mathscr{A}}$ to $\mathscr{P}(\mathscr{A})$, defined by (2.4), is increasing tow-sided, that is, for any ψ_1 and ψ_2 in $\Phi_{\mathscr{A}}$, $\psi_1 < \psi_2 \Leftrightarrow \mathscr{K}(\psi_1) \subset \mathscr{K}(\psi_2)$.* □

PROOF: 1. The only if part: $\psi_1(X) \subset \psi_2(X)$ $(X \in \mathscr{A})$ implies that for any $X \in \mathscr{K}(\psi_1)$, from (2.4), $o \in \psi_1(X) \subset \psi_2(X)$, which proves, from (2.4), that $X \in \mathscr{K}(\psi_2)$ and, consequently, $\mathscr{K}(\psi_1) \subset \mathscr{K}(\psi_2)$.

2. The if part: let $X \in \mathscr{A}$ and $x \in \psi_1(X)$, then, by Property 2.1, $X \in (\mathscr{K}(\psi_1))_x \subset (\mathscr{K}(\psi_2))_x$, but this implies, by Property 2.1, that $x \in \psi_2(X)$ which proves that $\psi_1(X) \subset \psi_2(X)$. □

The posets $(\Phi_{\mathscr{A}}, <)$ and $(\mathscr{P}(\mathscr{A}), \subset)$ are complete lattices, hence all the above lemmas, relative to the mapping $\mathscr{K}(\cdot)$, can be resumed in the following lemma.

LEMMA 2.5 - *Let $\mathscr{A} \subset \mathscr{P}(E)$ be closed under translation. The mapping $\mathscr{K}(\cdot)$ from $\Phi_{\mathscr{A}}$ onto $\mathscr{P}(\mathscr{A})$, defined by (2.4), is a lattice-isomorphism.* □

PROOF: Lemmas 2.3 and 2.4 together are equivalent to say that $\mathscr{K}(\cdot)$ is a lattice-isomorphism (see Lemma 2 p. 24 in Birkhoff (1967)). □

Let $\{\psi_i : i \in I\}$ be a family of t.i. mappings from $\mathscr{A}$ to $\mathscr{P}(E)$. The above Lemma 2.5 says in particular that

$$\mathscr{K}(\sqcup \{\psi_i : i \in I\}) = \cup \{\mathscr{K}(\psi_i) : i \in I\}$$

and

$$\mathscr{K}(\sqcap \{\psi_i : i \in I\}) = \cap \{\mathscr{K}(\psi_i) : i \in I\}.$$

In other words, the kernel of the supremum (under $<$) of a family of t.i. mappings is the union (or supremum under $\subset$) of the set of the corresponding kernels.

Before ending this chapter, important t.i. mappings are given and some duality properties recalled.

Let $\mathscr{A} \subset \mathscr{P}(E)$ be closed under translation. If ψ_1 and ψ_2 are two t.i. mappings, respectively, from $\mathscr{A}$ to $\mathscr{P}(E)$ and from $\mathscr{P}(E)$ to $\mathscr{P}(E)$, then ψ, the composition of ψ_1 and ψ_2, that is, $\psi = \psi_2 \circ \psi_1$, is a t.i. mapping from $\mathscr{A}$ to $\mathscr{P}(E)$. This can be seen as follows: for any $X \in \mathscr{A}$ and $h \in E$,

$$\psi(X_h) = \psi_2(\psi_1(X_h)),$$

by t.i. definition, $$= \psi_2((\psi_1(X))_h),$$

by t.i. definition, $$= (\psi_2(\psi_1(X)))_h,$$

$$= (\psi(X))_h.$$

Let $\mathscr{A} \subset \mathscr{P}(E)$, and $C_{\mathscr{A}}$ the mapping from $\mathscr{A}$ to $\mathscr{P}(E)$ defined by

$$C_{\mathscr{A}}X = \left\{x \in E:\ x \notin X\right\}, \tag{2.6}$$

for any $X \in \mathscr{A}$. $C_{\mathscr{P}(E)}X$, the complementary set of X, is denoted X^c. Let $\mathscr{A}^*$ be the image of $\mathscr{A}$ by $C_{\mathscr{A}}$, that is,

$$\mathscr{A}^* = C_{\mathscr{A}}\mathscr{A} = \left\{X \in \mathscr{P}(E):\ X^c \in \mathscr{A}\right\}.$$

In particular, $\mathscr{P}(E)^* = \mathscr{P}(E)$.

Let $\mathscr{A} \subset \mathscr{P}(E)$ be closed under translation, then, from (2.6), for any $X \in \mathscr{A}$,

$$C_{\mathscr{A}}X = \left\{x \in E:\ X \in \left\{Y \in \mathscr{A}:\ o \notin Y\right\}_x\right\}.$$

Therefore, by identifying with expression (2.5),

$\mathscr{E} = \left\{Y \in \mathscr{A} : o \in Y\right\}$ and, by applying Property 2.3, $C_{\mathscr{A}}$ is a t.i. mapping.

The t.i. property for $C_{\mathscr{A}}$ implies that $\mathscr{A}^*$ is closed under translation, for

$$\mathscr{A}^*_h = (C_{\mathscr{A}}\mathscr{A})_h = C_{\mathscr{A}}\mathscr{A}_h = C_{\mathscr{A}}\mathscr{A} = \mathscr{A}^*.$$

Let $\mathscr{A}_1$ and $\mathscr{A}_2 \subset \mathscr{P}(E)$ be closed under translation. Let ψ_1 and ψ_2 be two mappings from, respectively, $\mathscr{A}_1$ and $\mathscr{A}_2$ to $\mathscr{P}(E)$. ψ_1 and ψ_2 are said to be *dual* iff $\mathscr{A}_1 = \mathscr{A}_2^*$ or, equivalently, $\mathscr{A}_2 = \mathscr{A}_1^*$ and $\psi_1 = C_{\mathscr{P}(E)} \circ \psi_2 \circ C_{\mathscr{A}_1}$ or, equivalently, $\psi_2 = C_{\mathscr{P}(E)} \circ \psi_1 \circ C_{\mathscr{A}_2}$. In other words ψ_1 and ψ_2 are dual iff

$$\psi_1(X) = (\psi_2(X^c))^c \quad (X \in \mathscr{A}_1).$$

The *dual mapping* of a mapping ψ from $\mathscr{A} \subset \mathscr{P}(E)$ to $\mathscr{P}(E)$, denoted ψ^*, is defined by

$$\psi^* = C_{\mathscr{P}(E)} \circ \psi \circ C_{\mathscr{A}^*}.$$

Hence, ψ_1 and ψ_2 are dual iff $\psi_1 = \psi_2^*$ or, equivalently, $\psi_2 = \psi_1^*$.

If ψ is a t.i. mapping then, by composition of t.i. mappings, ψ^* is also a t.i. mapping.

Furthermore, if ψ_1 and ψ_2 are two mappings from $\mathscr{A}$ to $\mathscr{P}(E)$, then, by Morgan's law, the dual of their supremum, under $<$, is the infimum, under $<$, of their dual, that is,

$$(\psi_1 \sqcup \psi_2)^* = \psi_1^* \sqcap \psi_2^*.$$

PROPERTY 2.4 - *Let* $\mathcal{A}_1$ *and* $\mathcal{A}_2 \subset \mathcal{P}(E)$ *be closed under translation. Let* ψ_1 *and* ψ_2 *be two t.i. mappings from, respectively,* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *to* $\mathcal{P}(E)$ *and let* $\mathcal{K}(\psi_1)$ *and* $\mathcal{K}(\psi_2)$ *be their respective kernel, defined by* (2.4), *then* $\psi_1 = \psi_2$ iff $\mathcal{A}_1 = \mathcal{A}_2^*$ and $X \in \mathcal{K}(\psi_1) \Leftrightarrow X^c \notin \mathcal{K}(\psi_2)$ $(X \in \mathcal{A}_1)$. □

PROOF: 1. For any $X \in \mathcal{A}_1$,

from (2.4), $X \in \mathcal{K}(\psi_1) \quad \Leftrightarrow o \in \psi_1(X)$,

by dual definition, $\quad \Leftrightarrow o \in (\psi_2(X^c))^c$,

$\Leftrightarrow o \notin \psi_2(X^c)$,

from (2.4), $\quad \Leftrightarrow X^c \notin \mathcal{K}(\psi_2)$.

2. For any $X \in \mathcal{A}_1$,

by Property 2.2, $\psi_1(X) = \left\{x \in E:\ X \in (\mathcal{K}(\psi_1))_x\right\}$,

by assumption, $\quad = \left\{x \in E:\ X^c \notin (\mathcal{K}(\psi_2))_x\right\}$,

$= \left\{x \in E:\ X^c \in (\mathcal{K}(\psi_2))_x\right\}^c$,

by Property 2.2, $\quad = (\psi_2(X^c))^c$. □

The *Minkowski addition* $\oplus$ (Minkowski, 1903; Hadwiger, 1957) is defined in $\mathcal{P}(E)$ by

$$A \oplus B = \left\{x \in E:\ x = a + b,\ a \in A \text{ and } b \in B\right\}.$$

Let $A \in \mathcal{P}(E)$, the *symmetrical set* of A, denoted $\check{A}$, is:

$$\check{A} = \left\{x \in E:\ -x \in A\right\}.$$

Let $\mathcal{A} \subset \mathcal{P}(E)$, $X \in \mathcal{A}$ and $A \in \mathcal{P}(E)$, the set $X \oplus \check{A}$ is called, by Matheron (1975), the *dilation* of X by the *structural element* A. For Haralick et al. (1987) and Giardina and Dougherty (1988) the dilation of X by A is simply $X \oplus A$. The set $X \oplus \check{A}$ can be expressed in the form:

$$X \oplus \check{A} = \left\{x \in E:\ X \cap A_x \neq \emptyset\right\}. \tag{2.7}$$

The mapping $\cdot \oplus \check{A}$ from $\mathcal{A}$ to $\mathcal{P}(E)$ is called the *dilation* by A.

The dual mapping of $\cdot \oplus \check{A}$ from $\mathcal{A}$ to $\mathcal{P}(E)$ is a mapping from $\mathcal{A}^*$ to $\mathcal{P}(E)$, denoted $\cdot \ominus \check{A}$ and called, by Matheron (1975), the *erosion* by A. For Haralick (1987) and Giardina and Dougherty (1988) the definition of erosion is the same, but Haralick denotes it simply as $\cdot \ominus A$. In other words, the symbol $\ominus$ has another meaning and it can be observed that Haralick's dilation and erosion are not dual, in the sense given above. The set $X \ominus \check{A}$ is called the *erosion* of X by the *structural element* A and can be expressed in the form:

$$X \ominus \check{A} = \left\{x \in E:\ X^c \cap A_x \neq \emptyset\right\}^c,$$

$$= \left\{x \in E:\ A_x \subset X\right\}. \tag{2.8}$$

The dual property leads to the formula:

$$(X \oplus A)^c = X^c \ominus A \quad (X \in \mathcal{A})$$

(with Haralick's dilation and erosion definition the corresponding formula is: $(X \oplus A)^c = X^c \ominus \check{A}$).

Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation and $A \in \mathcal{P}(E)$, from (2.7), for any $X \in \mathcal{A}$,

$$X \oplus \check{A} = \left\{x \in E:\ X \in \left\{Y \in \mathcal{A}:\ Y \cap A \neq \emptyset\right\}_x\right\}.$$

Therefore, by identifying with expression (2.5), $\mathscr{C} = \left\{Y \in \mathscr{A}:\ Y \cap A \neq \emptyset\right\}$ and, applying Property 2.3, the mapping $\cdot \oplus \check{A}$ from $\mathscr{A}$ to $\mathscr{P}(E)$ is t.i.. By the duality property, the mapping $\cdot \ominus \check{A}$ from $\mathscr{A}$ to $\mathscr{P}(E)$ is also t.i..

From (2.7) and (2.8) the kernels of the dilation and the erosion from $\mathscr{A}$ to $\mathscr{P}(E)$ are:

$$\mathscr{K}(\cdot \oplus \check{A}) = \left\{X \in \mathscr{A}:\ X \cap A \neq \emptyset\right\}$$

and

$$\mathscr{K}(\cdot \ominus \check{A}) = \left\{X \in A:\ A \subset X\right\}.$$

The erosion and the complemented dilation of a set X are special cases of the general Hit-Miss mapping, due to Serra (1982). Let A and B be two disjoint subsets of E, then the *Hit-Miss transform* of X by the pair (A,B) is the set:

$$X \circledast (A,\ B) = \left\{x \in E:\ A_x \subset X \text{ and } B_x \subset X^c\right\}. \qquad (2.9)$$

From (2.8),

$$X \circledast (A,\ B) = (X \ominus \check{A}) \cap (X^c \ominus \check{B}) \qquad (X \in \mathscr{A}).$$

Let $\mathscr{A} \subset \mathscr{P}(E)$ be closed under translation. The mapping $\cdot \circledast (A,\ B)$ from $\mathscr{A}$ to $\mathscr{P}(E)$ is t.i., as infimum of two t.i. mappings, the second one being the composition of two t.i. mappings: the complementation and the erosion.

CHAPTER 3

REPRESENTATION THEOREMS FOR TRANSLATION INVARIENT MAPPINGS

3.1 - REPRESENTATION BY A SUPREMUM

For the moment, let E be any non empty set. Because of the nature of the t.i. mapping representation problem some definitions have to be made relatively to the elements of $\mathcal{P}(E) \times \mathcal{P}(E)$.

Let $\leq$ be the binary relation between pairs in $\mathcal{P}(E)^2$ defined by

$$(A_1, B_1) \leq (A_2, B_2) \text{ iff } A_1 \supset A_2 \text{ and } B_1 \subset B_2. \tag{3.1}$$

The relation $\leq$, defined by (3.1), is a partial order for $\mathcal{P}(E)^2$ (i.e., $\leq$ is reflexive, antisymmetric and transitive). The pairs $(E, \emptyset)$ and $(\emptyset, E)$ are, respectively, the smallest and greatest pairs in $\mathcal{P}(E)^2$. The supremum and infimum of two pairs (A_1, B_1) and (A_2, B_2) in $\mathcal{P}(E)^2$ always exist, are denoted, respectively, by $(A_1, B_1) \vee (A_2, B_2)$ and $(A_1, B_1) \wedge (A_2, B_2)$, and can be expressed as:

$$(A_1, B_1) \vee (A_2, B_2) = (A_1 \cap A_2, B_1 \cup B_2)$$

and

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cup A_2, B_1 \cap B_2).$$

From the above definitions,

$$\begin{aligned}(A_1, B_1) \leq (A_2, B_2) &\Leftrightarrow A_1 \supset A_2 \text{ and } B_1 \subset B_2 \\ &\Leftrightarrow A_1^c \cap A_2 = \emptyset \quad \text{and} \quad B_1^c \cup B_2 = E \\ &\Leftrightarrow (A_1^c, B_1^c) \vee (A_2, B_2) = (\emptyset, E),\end{aligned}$$

hence,

$$(A_1, B_1) \{ (A_2, B_2) \Leftrightarrow (A_1^c, B_1^c) \vee (A_2, B_2) = (\emptyset, E) \tag{3.2}$$

Furthermore $(\mathcal{P}(E)^2, \{)$ is a complete lattice.

Let $\mathcal{A}$ denote a subcollection of $\mathcal{P}(E)$, that is, $\mathcal{A} \subset \mathcal{P}(E)$, and $\mathfrak{H}_{\mathcal{A}}$ be the subset of $\mathcal{A}^2$ given by

$$\mathfrak{H}_{\mathcal{A}} = \left\{ \mathfrak{x} \in \mathcal{A}^2 \colon \exists\, X \in \mathcal{A} \colon (X, X) \{ \mathfrak{x} \right\},$$

or equivalently,

$$\mathfrak{H}_{\mathcal{A}} = \left\{ (A, B) \in \mathcal{A}^2 \colon A \subset B \right\}.$$

For $\mathfrak{H}_{\mathcal{A}}$ to be non empty, $\mathcal{A}$ must contain at least one pair (A, B) such that $A \subset B$ (e.g. $(\emptyset, E)$).

From (3.1), $\mathfrak{H}_{\mathcal{A}}$ is a dual ideal of $(\mathcal{A}^2, \{)$, that is, if $\mathfrak{y} \in \mathcal{A}^2$ and $\mathfrak{x} \in \mathfrak{H}_{\mathcal{A}}$ then $\mathfrak{x} \{ \mathfrak{y}$ implies $\mathfrak{y} \in \mathfrak{H}_{\mathcal{A}}$.

Let $\mathfrak{x} \in \mathfrak{H}_{\mathcal{P}(E)}$ and $\mathcal{X}_{\mathfrak{x}}^{\mathcal{A}}$ be the subcollection of $\mathcal{A}$ given by

$$\mathcal{X}_{\mathfrak{x}}^{\mathcal{A}} = \left\{ X \in \mathcal{A} \colon (X, X) \{ \mathfrak{x} \right\} \tag{3.3}$$

or, equivalently, with $\mathfrak{x} = (A, B)$,

$$\mathcal{X}_{(A, B)}^{\mathcal{A}} = \left\{ X \in \mathcal{A} \colon A \subset X \subset B \right\}.$$

If $\mathfrak{x}$ is restricted to be in $\mathfrak{H}_{\mathcal{A}}$ then $\mathcal{X}_{\mathfrak{x}}^{\mathcal{A}}$ is simply denoted $\mathcal{X}_{\mathfrak{x}}$ and called *closed interval* or *spindle* limited by $\mathfrak{x}$. $\mathfrak{x}$ is the *extremity pair* of the closed interval. If $\mathfrak{x} = (A, B)$, $\mathcal{X}_{\mathfrak{x}}$ is simply denoted $[A, B]$ and called the closed interval $[A, B]$.

The sets A and B are in $\mathcal{X}_{\mathfrak{x}}$ (i.e., $\mathcal{X}_{\mathfrak{x}}$ always exists) and are, respectively, the smallest and the greatest elements of $\mathcal{X}_{\mathfrak{x}}$. In particular, for any $X \in \mathcal{A}$, $\mathcal{X}_{(X,\ X)} = \left\{X\right\}$.

Let us now introduce one of the most important pieces for the t.i. mapping representation. Let $\mathcal{A} \subset \mathcal{P}(E)$ and $\mathfrak{R}_{\bullet}$ be the mapping from $\mathcal{P}(\mathcal{A})$ to $\mathcal{P}(\mathfrak{H}_{\mathcal{A}})$ defined by

$$\mathfrak{R}_{\mathcal{E}} = \left\{\mathfrak{x} \in \mathfrak{H}_{\mathcal{A}}\colon\ \mathcal{X}_{\mathfrak{x}} \subset \mathcal{E}\right\}, \tag{3.4}$$

for any $\mathcal{E} \in \mathcal{P}(\mathcal{A})$ or, equivalently, from (3.3),

$$\mathfrak{R}_{\mathcal{E}} = \left\{(A,\ B) \in \mathfrak{H}_{\mathcal{A}}\colon\ [AB] \subset \mathcal{E}\right\}.$$

$\mathfrak{R}_{\mathcal{E}}$ is the set of pairs (A, B) such that the closed intervals [A, B] are contained in $\mathcal{E}$ and it verifies:

$$X \in \mathcal{E} \text{ iff } (X,\ X) \in \mathfrak{R}_{\mathcal{E}}, \tag{3.5}$$

therefore, if $\mathcal{E}$ is non empty, $\mathfrak{R}_{\mathcal{E}}$,defined by (3.4), always exists. It verifies also:

$$\text{if } (A,\ B) \in \mathfrak{R}_{\mathcal{E}} \text{ then } A,\ B \in \mathcal{E}. \tag{3.6}$$

Let $\mathcal{A} \subset \mathcal{P}(E)$ and $\mathcal{R}^{\mathcal{A}}_{\bullet}$ be the mapping from $\mathcal{P}(\mathfrak{H}_{\mathcal{P}(E)})$ to $\mathcal{P}(\mathcal{A})$ defined by

$$\mathcal{R}^{\mathcal{A}}_{\mathfrak{C}} = \bigcup \left\{\mathcal{X}^{\mathcal{A}}_{\mathfrak{x}}\colon\ \mathfrak{x} \in \mathfrak{C}\right\}, \tag{3.7}$$

for any $\mathfrak{C} \in \mathcal{P}(\mathfrak{H}_{\mathcal{P}(E)})$.

The restriction of $\mathcal{R}^{\mathcal{A}}_{\bullet}$ to $\mathcal{P}(\mathfrak{H}_{\mathcal{A}})$ is denoted $\mathcal{R}_{\bullet}$. Such mapping is useful to study some properties of the mapping $\mathfrak{R}_{\bullet}$.

Let us derive now one of the most important results of this chapter.

PROPERTY 3.1 - *Let* $\mathscr{E} \subset \mathscr{A} \subset \mathscr{P}(E)$ *and* $\mathfrak{K}_{\mathscr{E}}$ *be the set defined by* (3.4), *then the collection* $\mathscr{R}_{\mathfrak{K}_{\mathscr{E}}}$, *defined by* (3.7, *with* $\mathfrak{C} = \mathfrak{K}_{\mathscr{E}}$), *is* $\mathscr{E}$, *that is*,

$$\mathscr{R}_{\mathfrak{K}_{\mathscr{E}}} = \mathscr{E}. \qquad \square$$

PROOF: From (3.4) and (3.7, with $\mathfrak{C} = \mathfrak{K}_{\mathscr{E}}$),

$$\mathscr{R}_{\mathfrak{K}_{\mathscr{E}}} = \bigcup \left\{\mathscr{X}_{\mathfrak{x}} : \mathfrak{x} \in \mathfrak{K}_{\mathscr{E}}\right\}$$

1. Let $X \in \mathscr{R}_{\mathfrak{K}_{\mathscr{E}}}$ then there exists $\mathfrak{x} \in \mathfrak{K}_{\mathscr{E}}$ such that $X \in \mathscr{X}_{\mathfrak{x}}$. From (3.4), for such $\mathfrak{x}$, $\mathscr{X}_{\mathfrak{x}} \subset \mathscr{E}$, hence $X \in \mathscr{E}$ and, consequently, $\mathscr{R}_{\mathfrak{K}_{\mathscr{E}}} \subset \mathscr{E}$.

2. Let $X \in \mathscr{E}$ then from (3.5), $\mathfrak{x} = (X, X) \in \mathfrak{K}_{\mathscr{E}}$; on the other hand, $Y \in \mathscr{X}_{(Y,\ Y)}$ for any $Y \in \mathscr{A}$, therefore, $X \in \mathscr{X}_{\mathfrak{x}}$ with $\mathfrak{x} \in \mathfrak{K}_{\mathscr{E}}$, hence $X \in \bigcup \left\{\mathscr{X}_{\mathfrak{y}} : \mathfrak{y} \in \mathfrak{K}_{\mathscr{E}}\right\}$ and, consequently, $\mathscr{E} \subset \mathscr{R}_{\mathfrak{K}_{\mathscr{E}}}$. $\square$

The Property 3.1 is, exactly, what is needed to derive, in the next section, the representation theorem.

This property gives also more insight on the mapping $\mathfrak{K}_{\bullet}$ since it proves that it is injective (see Property 6.3 p. 14 Dugundji, 1966), that is, the set of pairs $\mathfrak{K}_{\mathscr{E}}$ caracterizes, uniquely, the collection $\mathscr{E}$. On the other hand, a counter example can be given showing the existence, for a given $\mathscr{A}$, of a subset $\mathfrak{C}$ of $\mathfrak{H}_{\mathscr{A}}$ such that:

$$\mathfrak{K}_{\mathscr{R}_{\mathfrak{C}}} \neq \mathfrak{C}.$$

Together with Property 3.1, this proves that the above mapping $\mathfrak{R}_{\bullet}$ is not surjective.

The counter example can be build in the following way: let $A \in \mathcal{P}(E)$ and $\mathscr{C}_A$ be the collection defined by

$$\mathscr{C}_A = \left\{X \in \mathcal{P}(E):\ X = A_x \text{ and } x \in E\right\},$$

in other words, $\mathscr{C}_A$ contains A and is closed under translation. In particular $\mathscr{C}_{\emptyset} = \left\{\emptyset\right\}$ and $\mathscr{C}_E = \left\{E\right\}$. Let $\mathscr{A} = \mathscr{C}_{\emptyset} + \mathscr{C}_A + \mathscr{C}_B + \mathscr{C}_E$ [1] with $A \subset B \subset E$ and $A \neq E$. The set $\mathscr{A}$ is also closed under translation (this is a necessary condition to build t.i. mappings which domain is $\mathscr{A}$).

Let $\mathfrak{C}$ be the set $\left\{(A,\ E)\right\}$, from (3.7), $\mathcal{R}_{\mathfrak{C}} = [A,\ E]$ and, from (3.4),

$$\mathfrak{R}_{[A,\ E]} = \left\{(A,\ X) \in \mathfrak{H}_{\mathscr{A}}:\ X \in \mathscr{C}_B\right\} + \left\{(B,\ B),\ (B,\ E),\ (E,\ E)\right\}$$

which contains, in the proper sense, $\left\{(A,\ E)\right\}$, that is, $\mathfrak{C}$. In other words, in this example

$$\mathfrak{R}_{\mathcal{R}_{\mathfrak{C}}} \neq \mathfrak{C}.$$

PROPERTY 3.2 - *Let* $\mathscr{C}_1$ *and* $\mathscr{C}_2 \subset \mathscr{A} \subset \mathcal{P}(E)$ *and* $\mathfrak{R}_{\bullet}$ *be the mapping defined by* (3.4), *then*

$$\mathfrak{R}_{\mathscr{C}_1} \cap \mathfrak{R}_{\mathscr{C}_2} = \mathfrak{R}_{\mathscr{C}_1 \cap \mathscr{C}_2}. \qquad \square$$

PROOF: From (3.4),

$$\mathfrak{x} \in \mathfrak{R}_{\mathscr{C}_1} \cap \mathfrak{R}_{\mathscr{C}_2} \Leftrightarrow \mathcal{X}_{\mathfrak{x}} \subset \mathscr{C}_1 \text{ and } \mathcal{X}_{\mathfrak{x}} \subset \mathscr{C}_2,$$

---

[1] *+ stands for the union of disjoint sets.*

$$\Leftrightarrow \mathcal{X}_{\mathfrak{x}} \subset \mathcal{E}_1 \cap \mathcal{E}_2,$$

from (3.4), $$\Leftrightarrow \mathfrak{x} \in \mathfrak{K}_{\mathcal{E}_1 \cap \mathcal{E}_2}. \qquad \square$$

The above property shows that $\mathfrak{K}_{\bullet}$ is a meet-morphism. Actually, $\mathfrak{K}$ is *not* a join-morphism since, usually, just the following holds:

$$\mathfrak{K}_{\mathcal{E}_1} \cup \mathfrak{K}_{\mathcal{E}_2} \subset \mathfrak{K}_{\mathcal{E}_1 \cup \mathcal{E}_2}.$$

From now on, the set E is the Abelian group of Chapter 2.

Let $\mathscr{A} \subset \mathcal{P}(E)$ be closed under translation and $\Phi_{\mathscr{A}}$ be the set of t.i. mappings from $\mathscr{A}$ to $\mathcal{P}(E)$ (see Chapter 2).

In order to derive a representation for ψ, a mapping $\mathfrak{R}(\cdot)$ is now defined as the composition of $\mathcal{K}(\cdot)$ and $\mathfrak{K}_{\bullet}$, defined, respectively, by (2.4) and (3.4), that is,

$$\mathfrak{R}(\cdot) = \mathfrak{K}_{\bullet} \circ \mathcal{K}(\cdot).$$

In other words, $\mathfrak{R}(\cdot)$ is the mapping from $\Phi_{\mathscr{A}}$ to $\mathcal{P}(\mathfrak{H}_{\mathscr{A}})$ defined by

$$\mathfrak{R}(\psi) = \mathfrak{K}_{\mathcal{K}(\psi)}, \tag{3.8}$$

for any $\psi \in \Phi_{\mathscr{A}}$, or, equivalently,

$$\mathfrak{R}(\psi) = \left\{\mathfrak{x} \in \mathfrak{H}_{\mathscr{A}}:\ \mathcal{X}_{\mathfrak{x}} \subset \mathcal{K}(\psi)\right\},$$

or, from (3.3),

$$\mathfrak{R}(\psi) = \left\{\mathfrak{x} \in \mathfrak{H}_{\mathscr{A}}:\ (X,\ X)\ \{\ \mathfrak{x} \to X \in \mathcal{K}(\psi) \quad (X \in \mathscr{A})\right\}.$$

Some of the properties of the previous section can now be applied to the case of t.i. mappings.

PROPERTY 3.3 - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation, ψ be a t.i. mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ and let $\mathcal{K}(\psi)$ and $\mathfrak{K}(\psi)$ be the sets defined, respectively, by (2.4) and (3.8) and $\mathcal{R}_\bullet$ be the mapping defined by (3.8), then*

$$\mathcal{K}(\psi) = \mathcal{R}_{\mathfrak{K}(\psi)}. \qquad \square$$

PROOF: By Property 3.1 (with $\mathcal{E} = \mathcal{K}(\psi)$),

$$\mathcal{K}(\psi) = \mathcal{R}_{\mathfrak{K}_{\mathcal{K}(\psi)}},$$

from (3.8),

$$= \mathcal{R}_{\mathfrak{K}(\psi)}. \qquad \square$$

It has been seen in Chapter 2 that the infimum of two t.i. mappings is also a t.i. mapping. The following property about $\mathfrak{K}(\cdot)$ will be used in Chapter 6.

PROPERTY 3.4 ($\mathfrak{K}(\cdot)$ is a meet-morphism) - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation, ψ_1 and ψ_2 be two t.i. mappings from $\mathcal{A}$ to $\mathcal{P}(E)$ and let $\mathfrak{K}(\cdot)$ be the mapping defined by (3.8), then*

$$\mathfrak{K}(\psi_1 \sqcap \psi_2) = \mathfrak{K}(\psi_1) \cap \mathfrak{K}(\psi_2). \qquad \square$$

PROOF: This is a consequence of Lemma 2.5 and Property 3.2, with $\mathcal{E}_1 = \mathcal{K}(\psi_1)$ and $\mathcal{E}_2 = \mathcal{K}(\psi_2)$. $\square$

A new elementary t.i. mapping is now introduced which plays, because of its kernel property, a fundamental role in the t.i. mapping representation.

For any pair $\mathfrak{X} = (A, B)$ in $\mathfrak{H}_{\mathcal{A}}$ and $x \in E$, let $\mathfrak{X}_x$ denote the pair (A_x, B_x). If $\mathcal{A}$ is closed under translation then $\mathfrak{X}_x \in \mathfrak{H}_{\mathcal{A}}$. Let $\mathfrak{X} \in \mathfrak{H}_{\mathcal{P}(E)}$ and $\cdot \circledast \mathfrak{X}$ be the mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ defined by

$$X \odot \mathfrak{X} = \left\{x \in E:\ (X,\ X) \between \mathfrak{X}_x\right\}, \tag{3.9}$$

for any $X \in \mathcal{A}$. Writing $\mathfrak{X} = (A,\ B)$, an equivalent expression is:

$$X \odot (A,\ B) = \left\{x \in E:\ A_x \subset X \subset B_x\right\} \quad (X \in \mathcal{A}).$$

PROPERTY 3.5 - *Let* $\mathcal{A} \subset \mathcal{P}(E)$ *be closed under translation,* $\mathfrak{X} \in \mathfrak{H}_{\mathcal{P}(E)}$ *and* $\mathcal{X}^{\mathcal{A}}_{\mathfrak{X}}$ *be the collection defined by* (3.3). *The mapping* $\cdot \odot \mathfrak{X}$ *from* $\mathcal{A}$ *to* $\mathcal{P}(E)$, *defined by* (3.9), *is t.i. and its kernel, defined by* (2.4), *is:*

$$\mathcal{K}(\cdot \odot \mathfrak{X}) = \mathcal{X}^{\mathcal{A}}_{\mathfrak{X}}. \qquad \square$$

PROOF: For any $\mathfrak{X} \in \mathfrak{H}_{\mathcal{P}(E)}$, $\mathcal{A} \subset \mathcal{P}(E)$ and $X \in \mathcal{A}$,

$$X \odot \mathfrak{X} = \left\{x \in E:\ (X,\ X) \between \mathfrak{X}_x\right\},$$

from (2.1),
$$= \left\{x \in E:\ (X_{-x},\ X_{-x}) \between \mathfrak{X}\right\},$$

from (3.3),
$$= \left\{x \in E:\ X_{-x} \in \mathcal{X}^{\mathcal{A}}_{\mathfrak{X}}\right\},$$

from (2.2),
$$= \left\{x \in E:\ X \in (\mathcal{X}^{\mathcal{A}}_{\mathfrak{X}})_x\right\}.$$

Therefore by identifying with expression (2.5), $\mathcal{E} = \mathcal{X}^{\mathcal{A}}_{\mathfrak{X}}$, and, by applying Property 2.3, $\cdot \odot \mathfrak{X}$ is t.i. and its kernel is:

$$\mathcal{K}(\cdot \odot \mathfrak{X}) = \mathcal{X}^{\mathcal{A}}_{\mathfrak{X}}. \qquad \square$$

Writing $\mathfrak{X} = (A,\ B)$, an equivalent expression for the kernel of $\cdot \odot \mathfrak{X}$ is:

$$\mathcal{K}(\cdot \odot (A,\ B)) = \left\{X \in \mathcal{A}:\ A \subset X \subset B\right\} = AB.$$

On the other hand, for any $\mathfrak{x} \in \mathfrak{H}_{\mathcal{P}(E)}$, from (3.8),

$$\mathfrak{K}(\cdot \odot \mathfrak{x}) = \left\{ \mathfrak{y} \in \mathfrak{H}_{\mathcal{A}} : \ (X, X) \ \{ \ \mathfrak{y} \rightarrow (X, X) \ \{ \ \mathfrak{x} \quad (X \in \mathcal{A}) \right\}$$

$$= \left\{ \mathfrak{y} \in \mathfrak{H}_{\mathcal{A}} : \ \mathfrak{y} \ \{ \ \mathfrak{x} \right\}.$$

Figure 3.1 shows one particular element of the kernel of $\cdot \odot (A, B)$ for two given subsets A and B of E.

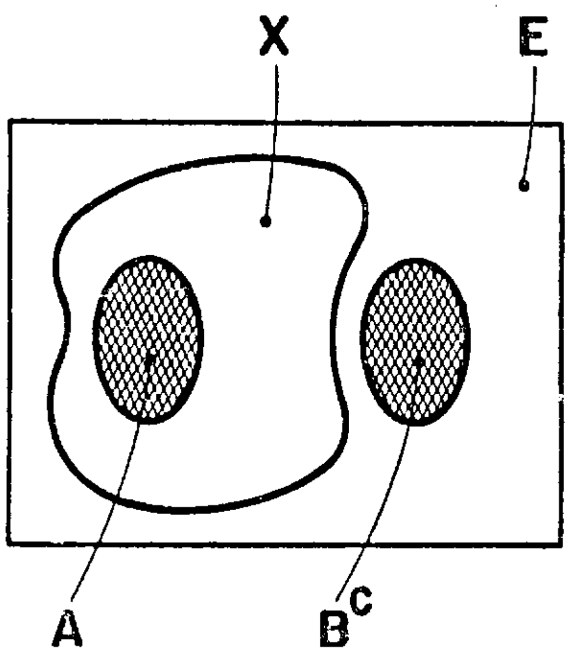

Fig. 3.1 - Example of a subset X belonging to the kernel of $\cdot \odot (A, B)$. X must contain A and miss B^c.

THEOREM 3.1 (Representation theorem) - *Let* $\mathcal{A} \subset \mathcal{P}(E)$ *be closed under translation,* $\cdot \odot \mathfrak{x}$ *be the mapping from* $\mathcal{A}$ *to* $\mathcal{P}(E)$, *defined by* (3.9), ψ *be a t.i. mapping from* $\mathcal{A}$ *to* $\mathcal{P}(E)$ *and* $\mathfrak{K}(\psi)$ *be the set defined by* (3.8), *then*

$$\psi = \bigsqcup \left\{ \cdot \odot \mathfrak{x} : \ \mathfrak{x} \in \mathfrak{K}(\psi) \right\} \qquad \square$$

PROOF: By Property 3.3 and from (3.7, with $\mathfrak{C} = \mathfrak{K}(\psi)$),

$$\mathcal{K}(\psi) = \bigcup \left\{\mathcal{X}_{\mathfrak{x}}:\ \mathfrak{x} \in \mathfrak{K}(\psi)\right\},$$

by Property 3.5,

$$= \bigcup \left\{\mathcal{K}(\cdot \bullet \mathfrak{x}):\ \mathfrak{x} \in \mathfrak{K}(\psi)\right\},$$

by Lemma 2.5,

$$\psi = \bigsqcup \left\{\cdot \bullet \mathfrak{x}:\ \mathfrak{x} \in \mathfrak{K}(\psi)\right\}$$ □

This result is important because it shows that the mapping $\cdot \bullet \mathfrak{x}$ is a prototype of any t.i. mapping. In other words, any t.i. mapping can be seen as the supremum of a family of elementary mappings $\cdot \bullet \mathfrak{x}$.

3.2 - REPRESENTATION BY AN INFIMUM

For the moment, let E be any non empty set. Let $\mathcal{A} \subset \mathcal{P}(E)$, $\mathfrak{x} \in \mathfrak{H}_{\mathcal{P}(E)}$ and $\mathcal{Y}_{\mathfrak{x}}^{\mathcal{A}}$ be the collection of all X in $\mathcal{A}$ such that $(X, X) \vee \mathfrak{x} \neq \mathbf{i}$, that is,

$$\mathcal{Y}_{\mathfrak{x}}^{\mathcal{A}} = \left\{X \in \mathcal{A}:\ (X, X) \vee \mathfrak{x} \neq \mathbf{i}\right\}, \tag{3.10}$$

where $\mathbf{i}$ stands for the pair $(\emptyset, E)$. If $\mathfrak{x}$ is restricted to be in $\mathfrak{H}_{\mathcal{A}}$ then $\mathcal{Y}_{\mathfrak{x}}^{\mathcal{A}}$ is simply denoted $\mathcal{Y}_{\mathfrak{x}}$.

PROPERTY 3.6 - *Let $\mathcal{A} \subset \mathcal{P}(E)$, $\mathfrak{x} \in \mathfrak{H}_{\mathcal{P}(E)}$ and $\mathcal{X}_{\mathfrak{x}}^{\mathcal{A}}$ and $\mathcal{Y}_{\mathfrak{x}}^{\mathcal{A}}$ be the collections defined, respectively, by (3.3) and (3.10), then, for any $X \in \mathcal{A}$,*

$$X \in \mathcal{X}_{\mathfrak{x}}^{\mathcal{A}} \Leftrightarrow X^c \notin \mathcal{Y}_{\mathfrak{x}^*}^{\mathcal{A}^*}$$

or, equivalently,

$$X^c \notin \mathcal{X}_{\mathfrak{x}^*}^{\mathcal{A}^*} \Leftrightarrow X \in \mathcal{Y}_{\mathfrak{x}}^{\mathcal{A}}.$$ □

PROOF: For any $X \in \mathscr{A}$,

from (3.3), $$X \in \mathscr{X}_{\mathfrak{x}}^{\mathscr{A}} \Leftrightarrow (X, X) \nmid \mathfrak{x},$$

from (3.2), $$\Leftrightarrow (X^c, X^c) \vee \mathfrak{x} = \mathfrak{i}$$

from (3.10), $$\Leftrightarrow X^c \notin \mathscr{Y}_{\mathfrak{x}}^{\mathscr{A}^*},$$

or, equivalently,

from (3.3), $$X^c \notin \mathscr{X}_{\mathfrak{x}}^{\mathscr{A}^*} \Leftrightarrow (X^c, X^c) \nmid \mathfrak{x},$$

from (3.2), $$\Leftrightarrow (X, X) \vee \mathfrak{x} \neq \mathfrak{i},$$

from (3.10), $$\Leftrightarrow \mathfrak{x} \in \mathfrak{Y}_{\mathfrak{x}}^{\mathscr{A}}. \qquad \square$$

From now on, the set E is the Abelian group of Chapter 2.

Let $\mathscr{A} \subset \mathscr{P}(E)$ be closed under translation. A new elementary t.i. mapping is now introduced.

Let $\mathfrak{x} \in \mathfrak{H}_{\mathscr{P}(E)}$ and let $\cdot \circledcirc \mathfrak{x}$ be the mapping from $\mathscr{A}$ to $\mathscr{P}(E)$ defined by

$$X \circledcirc \mathfrak{x} = \left\{x \in E:\ (X, X) \vee \mathfrak{x}_x \neq \mathfrak{i}\right\}, \qquad (3.11)$$

for any $X \in \mathscr{A}$. Writing $\mathfrak{x} = (A, B)$, an equivalent expression for (3.10) is:

$$X \circledcirc (A, B) = \left\{x \in E:\ X \cap A_x \neq \emptyset \text{ or } X \cup B_x \neq E\right\} \qquad (X \in \mathscr{A}).$$

PROPERTY 3.7 - *Let $\mathscr{A} \subset \mathscr{P}(E)$ be closed under translation, $\mathfrak{x}$ be in $\mathfrak{H}_{\mathscr{P}(E)}$ and $\mathscr{Y}_{\mathfrak{x}}^{\mathscr{A}}$ be the collection defined by* (3.10). *The mapping $\cdot \circledcirc \mathfrak{x}$ from $\mathscr{A}$ to $\mathscr{P}(E)$, defined by* (3.11), *is t.i. and its kernel, defined by* (2.4), *is:*

$$\mathcal{K}(\cdot \circledcirc \mathfrak{x}) = \mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}}. \qquad \square$$

PROOF: For any $\mathfrak{x} \in \mathfrak{H}_{\mathcal{P}(E)}$, $\mathcal{A} \subset \mathcal{P}(E)$ and $X \in \mathcal{A}$,

$$X \circledcirc \mathfrak{x} = \left\{x \in E:\ (X,\ X) \vee \mathfrak{x}_x \neq \mathfrak{i}\right\},$$

from (2.1),
$$= \left\{x \in E:\ (X_{-x},\ X_{-x}) \vee \mathfrak{x} \neq \mathfrak{i}\right\},$$

from (3.10),
$$= \left\{x \in E:\ X_{-x} \in \mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}}\right\},$$

from (2.2),
$$= \left\{x \in E:\ X \in (\mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}})_x\right\}.$$

Therefore, by identifying with expression (2.5), $\mathcal{B} = \mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}}$, and, by applying Property 2.3, $\cdot \circledcirc \mathfrak{x}$ is t.i. and its kernel is:

$$\mathcal{K}(\cdot \circledcirc \mathfrak{x}) = \mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}}. \qquad \square$$

Writing $\mathfrak{x} = (A,\ B)$, an equivalent expression for the kernel of $\cdot \circledcirc \mathfrak{x}$ is:

$$\mathcal{K}(\cdot \circledcirc (A,\ B)) = \left\{X \in \mathcal{A}:\ X \cap A \neq \emptyset \text{ or } X \cup B \neq E\right\}.$$

On the other hand, for any $\mathfrak{x} \in \mathfrak{H}_{\mathcal{P}(E)}$, from (3.8),

$$\mathfrak{K}(\cdot \circledcirc \mathfrak{x}) = \left\{\mathfrak{y} \in \mathfrak{H}_{\mathcal{A}}:\ (X,\ X) \nleq \mathfrak{y} \Rightarrow (X,\ X) \vee \mathfrak{x} \neq \mathfrak{i} \quad (X \in \mathcal{A})\right\}$$

$$= \left\{\mathfrak{y} \in \mathfrak{H}_{\mathcal{A}}:\ \mathfrak{y} \vee \mathfrak{x} \neq \mathfrak{i}\right\}.$$

Figure 3.2 shows two particular elements of the kernel of $\cdot \circledcirc (A,\ B)$ for two given subsets A and B of E.

Let $\mathfrak{X} \in \mathfrak{H}_{\mathcal{P}(E)}$ and $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation. By Properties 3.5 and 3.7, the kernels of $\cdot \oplus \mathfrak{X}$ and $\cdot \otimes \mathfrak{X}$ from, respectively, $\mathcal{A}$ and $\mathcal{A}^*$ to $\mathcal{P}(E)$ are $\mathcal{X}^{\mathcal{A}}_{\mathfrak{X}}$ and $\mathcal{Y}^{\mathcal{A}^*}_{\mathfrak{X}}$. Therefore, by Properties 2.4, and 3.6, $\cdot \oplus \mathfrak{X}$ and $\cdot \otimes \mathfrak{X}$ are dual mappings. Making $\mathcal{A} = \mathcal{P}(E)$, this leads to the formula:

$$(X \otimes \mathfrak{X})^c = X^c \oplus \mathfrak{X} \quad (X \in \mathcal{P}(E)). \tag{3.12}$$

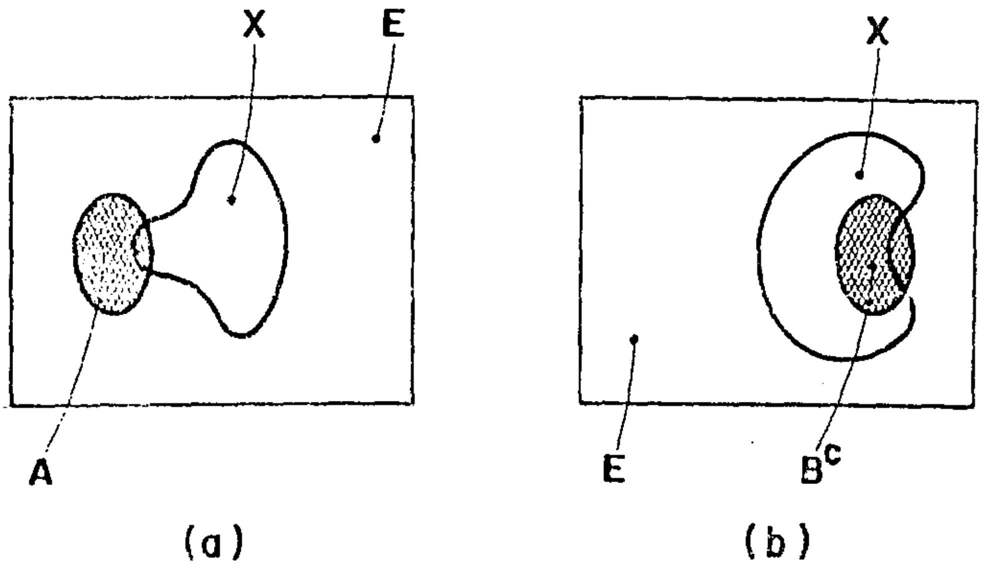

Fig. 3.2 - Example of two subsets X belonging to the kernel of $\cdot \otimes (A, B)$. X must hit A (a) or not contain B^c (b).

Let $\mathcal{A} \subset \mathcal{P}(E)$ and let $\mathcal{S}^{\mathcal{A}}_{\bullet}$ be the mapping from $\mathcal{P}(\mathfrak{H}_{\mathcal{P}(E)})$ to $\mathcal{P}(\mathcal{A})$ defined by

$$\mathcal{S}^{\mathcal{A}}_{\mathfrak{C}} = \bigcap \left\{ \mathcal{Y}^{\mathcal{A}}_{\mathfrak{X}} : \mathfrak{X} \in \mathfrak{C} \right\}, \tag{3.13}$$

for any $\mathfrak{C} \in \mathcal{P}(\mathfrak{H}_{\mathcal{P}(E)})$. The restriction of $\mathcal{S}^{\mathcal{A}}_{\bullet}$ to $\mathcal{P}(\mathfrak{H}_{\mathcal{A}})$ is denoted $\mathcal{S}_{\bullet}$.

PROPERTY 3.8 - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation, ψ be a t.i. mapping from $\mathcal{A}$ to $\mathcal{P}(E)$, ψ^* be its dual, $\mathcal{K}(\psi)$ and*

$\mathfrak{R}(\psi^*)$ *be the sets defined, respoectively by* (2.4) *and* (3.8) *and* $\mathcal{S}^{\mathscr{A}}_{\bullet}$ *be the mapping defined by* (3.13) *then*

$$\mathcal{K}(\psi) = \mathcal{S}^{\mathscr{A}}_{\mathfrak{R}(\psi^*)}. \qquad \square$$

PROOF: Let $\mathfrak{C} = \mathfrak{R}(\psi^*)$, for any $X \in \mathscr{A}$, by Property 2.4,

$$X \in \mathcal{K}(\psi) \Leftrightarrow X^c \notin \mathcal{K}(\psi^*),$$

by Property 3.3, $\quad \Leftrightarrow X^c \notin \mathcal{R}_{\mathfrak{C}}$,

because $\mathfrak{C} \in \mathcal{P}(\mathfrak{H}_{\mathscr{A}^*})$, $\quad \Leftrightarrow X^c \notin \mathcal{R}^{\mathscr{A}^*}_{\mathfrak{C}}$,

from (3.7), $\quad \Leftrightarrow X^c \notin \mathcal{X}^{\mathscr{A}^*}_{\mathfrak{x}}$ for any $\mathfrak{x} \in \mathfrak{C}$,

by Property 3.7, $\quad \Leftrightarrow X \in \mathcal{Y}^{\mathscr{A}}_{\mathfrak{x}}$ for any $\mathfrak{x} \in \mathfrak{C}$,

from (3.13), $\quad \Leftrightarrow X \in \mathcal{S}_{\mathfrak{C}}$. $\quad \square$

It can be observed that $\mathfrak{R}(\cdot)$, in Property 3.8, is a mapping from $\Phi_{\mathscr{A}^*}$ to $\mathcal{P}(\mathfrak{H}_{\mathscr{A}^*})$ and if $\mathscr{A} = \mathscr{A}^*$ then $\mathfrak{R}(\psi^*) \in \mathcal{P}(\mathfrak{H}_{\mathscr{A}})$, and $\mathcal{K}(\psi) = \mathcal{S}_{\mathfrak{R}(\psi^*)}$.

THEOREM 3.2 (Dual representation theorem) - *Let* $\mathscr{A} \subset \mathcal{P}(E)$ *be closed under translation,* $\cdot \circledcirc \mathfrak{x}$ *be the mapping from* $\mathscr{A}$ *to* $\mathcal{P}(E)$, *defined by* (3.11), ψ *be a t.i. mapping from* $\mathscr{A}$ *to* $\mathcal{P}(E)$ *and* $\mathfrak{R}(\psi^*)$ *be the set defined by* (3.8), *where* ψ^* *is the dual mapping of* ψ, *then*

$$\psi = \prod \left\{ \cdot \circledcirc \mathfrak{x} : \mathfrak{x} \in \mathfrak{R}(\psi^*) \right\}. \qquad \square$$

PROOF: By Property 3.8 and from (3.13, with $\mathfrak{C} = \mathfrak{R}(\psi^*)$),

$$\mathcal{K}(\psi) = \bigcap \left\{ \mathcal{Y}^{\mathscr{A}}_{\mathfrak{x}} : \mathfrak{x} \in \mathfrak{R}(\psi^*) \right\},$$

by Property 3.7,

$$= \bigcap \left\{ \mathcal{K}(\cdot \circledcirc \mathfrak{X}) : \mathfrak{X} \in \mathfrak{K}(\psi^*) \right\},$$

by Lemma 2.5,

$$\psi = \prod \left\{ \cdot \circledcirc \mathfrak{X} : \mathfrak{X} \in \mathfrak{K}(\psi^*) \right\}. \qquad \square$$

This result is important because it gives an alternative way to represent ψ. To represent ψ one form or the other is chosen, depending on which of the set $\mathfrak{K}(\psi)$ or $\mathfrak{K}(\psi^*)$ is simpler.

The sets $X \circledast \mathfrak{X}$ and $X \circledcirc \mathfrak{X}$ appearing in the representation of a t.i. mapping in Theorems 3.1 and 3.2 can be written, as it can be seen below, respectively, in terms of intersection (this is the reason for using the symbol $\circledast$) of erosions, and of union (this is the reason for using the symbol $\circledcirc$) of dilations.

Let $A, B \in \mathcal{P}(E)$ and let $X \circledast (A, B)$ and $X \circledcirc (A, B)$ be the two sets given by, respectively, (3.9, with $\mathfrak{X} = (A, B)$) and (3.11, with $\mathfrak{X} = (A, B)$), then

$$X \circledast (A, B) = (X \ominus \check{A}) \cap (X^c \ominus \check{B}^c) \quad (X \in \mathcal{A}) \qquad (3.14)$$

$$X \circledcirc (A, B) = (X \oplus \check{A}) \cup (X^c \oplus \check{B}^c) \quad (X \in \mathcal{A}). \qquad (3.15)$$

This can be proved in the following way:

from (3.9),

$$X \circledast (A, B) = \left\{ x \in E : A_x \subset X \text{ and } X \subset B_x \right\},$$

$$= \left\{ x \in E : A_x \subset X \text{ and } B_x^c \subset X^c \right\},$$

$$= \left\{ x \in E : A_x \subset X \right\} \cap \left\{ x \in E : B_x^c \subset X^c \right\},$$

from (2.8),

$$= (X \ominus \check{A}) \cap (X^c \ominus \check{B}^c).$$

From (3.12),

$$X \circledcirc (A, B) = (X^c \odot (A, B))^c,$$

$$= ((X^c \ominus \check{A}) \cap (X \ominus \check{B}^c))^c,$$

by Morgan's law and duality,

$$= (X \oplus \check{A}) \cup (X^c \oplus \check{B}^c).$$

In terms of the Hit-Miss transform of X, from (2.9) and (3.9),

$$X \odot (A, B) = X \otimes (A, B^c) \quad (X \in \mathscr{A}).$$

CHAPTER 4

INCREASING, DECREASING AND INF-SEPARABLE TRANSLATION INVARIANT MAPPINGS

In this chapter E is the Abelian group of Chapter 2.

The objective of this chapter is to study the special cases of increasing, decreasing and inf-separable t.i. mappings and to show in the former case that the representation theorem, given by Matheron (1975), is a special case of Theorem 3.1. A mapping ψ from $\mathcal{A} \subset \mathcal{P}(E)$ to $\mathcal{P}(E)$ is said to be *increasing* iff

for any X and $Z \in \mathcal{A}$, $X \subset Z$ implies $\psi(X) \subset \psi(Z)$,

decreasing iff

for any Y and $Z \in \mathcal{A}$, $Z \subset Y$ implies $\psi(Y) \subset \psi(Z)$

and *inf-separable* or *spindle-shaped*[1] iff

for any X, Y and $Z \in \mathcal{A}$,

$X \subset Z \subset Y$ implies $\psi(X) \cap \psi(Y) \subset \psi(Z)$.

From these definitions, any increasing and decreasing mappings are inf-separable mappings:

$X \subset Z \subset Y$ implies $\psi(X) \cap \psi(Y) = \psi(X) \subset \psi(Z)$

if ψ is increasing,

$X \subset Z \subset Y$ implies $\psi(X) \cap \psi(Y) = \psi(Y) \subset \psi(Z)$

if ψ is decreasing. But the contrary is false.

[1] *The french word* "fuselé" *translated here by* "spindle-shaped" *has been suggested to the authors by G. Matheron.*

PROPERTY 4.1 - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation, ψ be a t.i. mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ and $\mathcal{K}(\psi)$ be its kernel, defined by (2.4), then*

1. *ψ is increasing iff $\mathcal{K}(\psi)$ is a dual ideal of $(\mathcal{A}, \subset)$ (i.e., if $X \in \mathcal{K}(\psi)$ and $Z \in \mathcal{A}$, then $X \subset Z$ implies that $Z \in \mathcal{K}(\psi)$);*

2. *ψ is decreasing iff $\mathcal{K}(\psi)$ is an ideal of $(\mathcal{A}, \subset)$ (i.e., if $Y \in \mathcal{K}(\psi)$ and $Z \in \mathcal{A}$, then $Z \subset Y$ implies that $Z \in \mathcal{K}(\psi)$);*

3. *ψ is inf-separable iff $\mathcal{K}(\psi)$ is such that if X and $Y \in \mathcal{K}(\psi)$ and $Z \in \mathcal{A}$, then $X \subset Z \subset Y$ implies that $Z \in \mathcal{K}(\psi)$.* □

PROOF: X and $Y \in \mathcal{K}(\psi)$ implies, from (2.4), that $o \in \psi(X)$ and $\psi(Y)$. Therefore, for any of the three types of t.i. mapping $o \in \psi(Z)$, i.e., from (2.4), $Z \in \mathcal{K}(\psi)$. Conversely,

1. Let $X \subset Z$ and $x \in \psi(X)$. By Property 2.1 $X \in (\mathcal{K}(\psi))_x$ and under the dual ideal assumption on $\mathcal{K}(\psi)$, $Z \in (\mathcal{K}(\psi))_x$, that is, by Property 2.1, $x \in \psi(Z)$.

2. Let $Z \subset Y$ and $x \in \psi(Y)$. By Property 2.1 $Y \in (\mathcal{K}(\psi))_x$ and under the ideal assumption on $\mathcal{K}(\psi)$, $Z \in (\mathcal{K}(\psi))_x$, that is, by Property 2.1, $x \in \psi(Z)$.

3. Let $X \subset Z \subset Y$ and $x \in \psi(X) \cap \psi(Y)$. By Property 2.1 X and $Y \in (\mathcal{K}(\psi))_x$ and under the assumption on $\mathcal{K}(\psi)$, $Z \in (\mathcal{K}(\psi))_x$, that is, by Property 2.1, $x \in \psi(Z)$. □

The kernels of increasing or decreasing mappings satisfy the property of the kernels of inf-separable mappings.

PROPERTY 4.2 - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation, ψ*

be a t.i. mapping from $\mathscr{A}$ *to* $\mathscr{P}(E)$ *and let* $\mathscr{K}(\psi)$ *and* $\mathfrak{R}(\psi)$ *be the sets defined, respectively, by (2.4) and (3.8). If* X *and* Y $\in \mathscr{K}(\psi)$, *and* Z $\in \mathscr{A}$, *then*

1. $X \subset Y$ *implies* $(X, Y) \in \mathfrak{R}(\psi)$ *iff* ψ *is inf-separable,*

2. $X \subset Z$ *implies* $(X, Z) \in \mathfrak{R}(\psi)$ *iff* ψ *is increasing,*

3. $Z \subset Y$ *implies* $(Z, Y) \in \mathfrak{R}(\psi)$ *iff* ψ *is decreasing.* □

PROOF: From (3.3), the statement $X \subset Z \subset Y$ implies that $Z \in \mathscr{K}(\psi)$ is equivalent to $[X, Y] \subset \mathscr{K}(\psi)$. Therefore, by Property 4.1, part 3, if any X and Y $\in \mathscr{K}(\psi)$, $X \subset Y$, then $[X, Y] \subset \mathscr{K}(\psi)$ iff ψ is inf-separable. Consequently, the statement $Z \in \mathscr{K}(\psi)$, $(X \subset Z)$, is equivalent to $[X, Z] \subset \mathscr{K}(\psi)$ and the statement $Z \in \mathscr{K}(\psi)$, $(Z \subset Y)$, is equivalent to $[Z, Y] \subset \mathscr{K}(\psi)$. Therefore, the result, part 2 and 3, follows by Property 4.1, part 1 and 2. □

PROPERTY 4.3 - *Let* $\mathscr{A} \subset \mathscr{P}(E)$ *be closed under translation,* ψ *be a t.i. mapping from* $\mathscr{A}$ *to* $\mathscr{P}(E)$ *and* $\mathscr{K}(\psi)$ *and* $\mathfrak{R}(\psi)$ *be the sets defined, respectively, by (2.4) and (3.8), then*

1. $\mathfrak{R}(\psi) = (\mathscr{K}(\psi) \times \mathscr{A}) \cap \mathfrak{H}_{\mathscr{A}}$ *iff* ψ *is increasing,*

2. $\mathfrak{R}(\psi) = (\mathscr{A} \times \mathscr{K}(\psi)) \cap \mathfrak{H}_{\mathscr{A}}$ *iff* ψ *is decreasing,*

3. $\mathfrak{R}(\psi) = (\mathscr{K}(\psi) \times \mathscr{K}(\psi)) \cap \mathfrak{H}_{\mathscr{A}}$ *iff* ψ *is inf-separable.* □

PROOF: If $(A, B) \in \mathfrak{R}(\psi)$, from (3.6, with $\mathscr{B} = \mathscr{K}(\psi)$) A and B $\in \mathscr{K}(\psi)$, that is, $\mathfrak{R}(\psi) \subset (\mathscr{K}(\psi) \times \mathscr{K}(\psi)) \cap \mathfrak{H}_{\mathscr{A}}$. Conversely, by Proposition 4.2,

1. with X = A and Z = B, $(A, B) \in (\mathscr{K}(\psi) \times \mathscr{A}) \cap \mathfrak{H}_{\mathscr{A}}$ implies that $(A, B) \in \mathfrak{R}(\psi)$ iff ψ is increasing,

2. with $Z = A$ and $Y = B$, $(A, B) \in (\mathscr{A} \times \mathscr{K}(\psi)) \cap \mathfrak{H}_{\mathscr{A}}$ implies that $(A, B) \in \mathfrak{K}(\psi)$ iff ψ is decreasing.

3. with $X = A$ and $Y = B$, $(A, B) \in (\mathscr{K}(\psi) \times \mathscr{K}(\psi)) \cap \mathfrak{H}_{\mathscr{A}}$ implies that $(A, B) \in \mathfrak{K}(\psi)$ iff ψ is inf-separable. $\square$

Let ψ be any t.i. mapping from $\mathscr{A}$ to $\mathscr{P}(E)$, $\mathfrak{K}(\psi)$ be the set defined by (3.8) and $\mathscr{K}^A(\psi)$ and $\mathscr{K}_B(\psi)$ be the collections defined by

$$\mathscr{K}^A(\psi) = \left\{X \in \mathscr{A}:\ (A, X) \in \mathfrak{K}(\psi)\right\} \tag{4.1}$$

and

$$\mathscr{K}_B(\psi) = \left\{X \in \mathscr{A}:\ (X, B) \in \mathfrak{K}(\psi)\right\}, \tag{4.2}$$

for any A and $B \in \mathscr{K}(\psi)$. From (3.6), if $(A, B) \in \mathfrak{K}(\psi)$ then A and $B \in \mathscr{K}(\psi)$, the kernel of ψ defined by (2.4). Therefore, by using $\mathscr{K}^A(\psi)$, the proposed representation for ψ becomes, by Theorem 3.1 and from (3.14),

$$\begin{aligned}\psi(X) &= \bigcup\left\{\bigcup\left\{(X \ominus \check{A}) \cap (X^c \ominus \check{B}^c):\ B \in \mathscr{K}^A(\psi)\right\}:\ A \in \mathscr{K}(\psi)\right\} \\ &= \bigcup\left\{(X \ominus \check{A}) \cap \bigcup\left\{X^c \ominus \check{B}^c:\ B \in \mathscr{K}^A(\psi)\right\}:\ A \in \mathscr{K}(\psi)\right\} \\ &\qquad (X \in \mathscr{A}).\end{aligned} \tag{4.3}$$

Comparing with Matheron's representation, the proposed representation for general t.i. mappings contains the extra term:

$$\bigcup\left\{X^c \ominus \check{B}^c:\ B \in \mathscr{K}^A(\psi)\right\}$$

which plays the role of a "correction term". Similarly, by using $\mathscr{K}_B(\psi)$,

$$\psi(X) = \bigcup \left\{(X^c \ominus \check{B}^c) \cap \bigcup \left\{X \ominus \check{A}:\ A \in \mathcal{K}_B(\psi)\right\}:\ B \in \mathcal{K}(\psi)\right\}$$

$$(X \in \mathcal{A}). \qquad (4.4)$$

THEOREM 4.1 - *Let* $\mathcal{A} \subset \mathcal{P}(E)$ *be closed under translation,* $\cdot \ominus \check{A}$ *be the erosion by* A *from* $\mathcal{A}$ *to* $\mathcal{P}(E)$, *defined by* (2.8), ψ *be a t.i. mapping from* $\mathcal{A}$ *to* $\mathcal{P}(E)$ *and* $\mathcal{K}(\psi)$ *be its kernel, defined by* (2.4), *then*

1. $\psi = \bigcup \left\{\cdot \ominus \check{A}:\ A \in \mathcal{K}(\psi)\right\}$ *if* ψ *is increasing,*

2. $\psi = \bigcup \left\{\cdot^c \ominus \check{B}^c:\ B \in \mathcal{K}(\psi)\right\}$ *if* ψ *is decreasing,*

3. $\psi = \bigcup \left\{(\cdot \ominus \check{A}) \cap (\cdot^c \ominus \check{B}^c):\ A,\ B \in \mathcal{K}(\psi)\right\}$ *if* ψ *is inf-separable.* □

PROOF: By Theorem 3.1 any t.i. mapping can be represented as in (4.3) and (4.4). Hence, for increasing (respectively, decreasing) t.i. mappings the result follows from (4.3) (respectively, from (4.4)) if it can be proved that, for any $X \in \mathcal{A}$ and $A \in \mathcal{K}(\psi)$,

$$(X \ominus \check{A}) \subset \bigcup \left\{X^c \ominus \check{B}^c:\ B \in \mathcal{K}^A(\psi)\right\}$$

(respectively, for any $X \in \mathcal{A}$ and $B \in \mathcal{K}(\psi)$,

$$(X^c \ominus \check{B}^c) \subset \bigcup \left\{X \ominus \check{A}:\ A \in \mathcal{K}_B(\psi)\right\}).$$

1. The increasing case: let $x \in X \ominus \check{A}$ or, equivalently, $A_x \subset X$ and let $Y = X_{-x}$ then $A \subset Y$ since $A \subset X_{-x}$. By Property 4.2, $(A,\ Y) \in \mathfrak{K}(\psi)$ and, from (4.1), $Y \in \mathcal{K}^A(\psi)$, but $Y = X_{-x}$ implies that $x \in X^c \ominus \check{Y}^c$, therefore,

$$x \in \bigcup \left\{X^c \ominus \check{B}^c:\ B \in \mathcal{K}^A(\psi)\right\}.$$

2. The decreasing case: let $x \in X^c \ominus \check{B}^c$ or, equivalently, $X \subset B_x$. and let $Y = X_{-x}$ then $Y \subset B$ since $X_{-x} \subset B$. By Property 4.2, $(Y, B) \in \mathfrak{K}(\psi)$ and, from (4.2), $Y \in \mathcal{K}_B(\psi)$, but $Y = X_{-x}$ implies that $x \in X \ominus \check{Y}$, therefore,

$$x \in \bigcup \left\{X \ominus \check{A}:\ A \in \mathcal{K}_B(\psi)\right\}.$$

For inf-separable t.i. mappings the result follows from (3.14) and by Theorem 3.1 and Property 4.3 since, for any $X \in \mathcal{A}$ and for any (A, B) belonging to $\mathcal{K}(\psi) \times \mathcal{K}(\psi)$ but not to $\mathfrak{H}_A$, $(X \ominus \check{A}) \cap (X^c \ominus \check{B}^c) = \emptyset$. □

The above representation for an increasing mapping in Theorem 4.1 is, exactly, Matheron's representation.

THEOREM 4.2 - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation. if ψ is an inf-separable mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ then there exist two mappings ψ_1* and *ψ_2 from A to $\mathcal{P}(E)$, respectively increasing and decreasing, such that $\psi = \psi_1 \sqcap \psi_2$. Conversely, if ψ_1 and ψ_2 are mappings from $\mathcal{A}$ to $\mathcal{P}(E)$, respectively increasing and decreasing, then the mapping $\psi = \psi_1 \sqcap \psi_2$ is an inf-separable mapping from $\mathcal{A}$ to $\mathcal{P}(E)$.* □

PROOF: 1. Let $\mathcal{K}(\psi)$ be the kernel of ψ defined by (2.8). Let

$$\mathcal{K}_1 = \left\{X \in \mathcal{A}:\ \exists\ A \in \mathcal{K}(\psi):\ A \subset X\right\}$$

and

$$\mathcal{K}_2 = \left\{Y \in \mathcal{A}:\ \exists\ B \in \mathcal{K}(\psi):\ Y \subset B\right\}.$$

For any $X \in \mathcal{K}_1$, there exists an $A \in \mathcal{K}(\psi)$ such that $A \subset X$, therefore, $\mathcal{K}_1$ is a dual ideal since for any $X \in \mathcal{K}_1$ and $Z \in \mathcal{A}$, $X \subset Z$, which means that $A \subset Z$, implies that $Z \in \mathcal{K}_1$.

For any $Y \in \mathcal{K}_2$, there exists a $B \in \mathcal{K}(\psi)$ such that $Y \subset B$, therefore, $\mathcal{K}_2$ is an ideal since for any $Y \in \mathcal{K}_2$ and $Z \in \mathcal{A}$, $Z \subset Y$, which means that $Z \subset B$, implies that $Z \in \mathcal{K}_2$.

Moreover, if $X \in \mathcal{K}(\psi)$ then $X \in \mathcal{K}_1$ and $\mathcal{K}_2$, therefore, $\mathcal{K}(\psi) \subset \mathcal{K}_1 \cap \mathcal{K}_2$; if $X \in \mathcal{K}_1 \cap \mathcal{K}_2$ then there exist A and $B \in \mathcal{K}(\psi)$ such that $A \subset X$ and $X \subset B$, by Property 4.1, under the assumption that ψ is inf-separable, $X \in \mathcal{K}(\psi)$, therefore $\mathcal{K}_1 \cap \mathcal{K}_2 \subset \mathcal{K}(\psi)$. That is $\mathcal{K}(\psi) = \mathcal{K}_1 \cap \mathcal{K}_2$. In other words, by Property 4.1, there exist ψ_1 and ψ_2, respectively, increasing and decreasing such that, by Lemma 2.5, $\psi = \psi_1 \sqcap \psi_2$.

2. If $\psi = \psi_1 \sqcap \psi_2$, then for any X, Y and Z such that $X \subset Z \subset Y$, $\psi_1(X) \subset \psi_1(Z)$ and $\psi_2(Y) \subset \psi_2(Z)$, therefore successively,

$$\psi_1(X) \cap \psi_2(Y) \subset \psi_1(Z) \cap \psi_2(Z),$$

$$(\psi_1(X) \cap \psi_2(X)) \cap (\psi_1(Y) \cap \psi_2(Y)) \subset \psi_1(Z) \cap \psi_2(Z)$$

and

$$\psi(X) \cap \psi(Y) \subset \psi(Z)$$

which proves that ψ is an inf-separable mapping. □

The above decomposition of an inf-separable mapping in terms of the infimum of increasing and decreasing mappings is not unique as it can be seen on a simple example throught the formula:

$$(X \oplus \langle x \rangle) \cap (X^c \oplus \langle x, y \rangle) = (X \oplus \langle x \rangle) \cap (X^c \oplus \langle y \rangle)$$

$$(X \in \mathcal{P}(E))$$

by taking x and y $\in$ E and $x \neq y$.

Finally a last property for inf-separable mappings is presented that will be used in Chapter 6.

PROPERTY 4.4 - *Let ψ_1 and ψ_2 be two t.i. mappings from $\mathcal{A}$ to $\mathcal{P}(E)$, respectively, increasing and decreasing and let $\psi = \psi_1 \sqcap \psi_2$. Let $\mathcal{K}(\psi_1)$ and $\mathcal{K}(\psi_2)$ be the kernels of ψ_1 and ψ_2, defined by (2.4), and $\mathfrak{R}(\psi)$ be defined by (3.8), then*

$$\mathfrak{R}(\psi) = (\mathcal{K}(\psi_1) \times \mathcal{K}(\psi_2)) \cap \mathfrak{H}_{\mathcal{A}}. \quad \square$$

PROOF: By Property 3.4,

$$\mathfrak{R}(\psi) = \mathfrak{R}(\psi_1) \cap \mathfrak{R}(\psi_2),$$

by Property 4.3, with $\psi = \psi_1$ increasing and $\psi = \psi_2$ decreasing,

$$= (\mathcal{K}(\psi_1) \times \mathcal{A}) \cap (\mathcal{A} \times \mathcal{K}(\psi_2)) \cap \mathfrak{H}_{\mathcal{A}},$$

$$= (\mathcal{K}(\psi_1) \times \mathcal{K}(\psi_2)) \cap \mathfrak{H}_{\mathcal{A}}. \quad \square$$

If $\psi = \psi_1 \sqcup \psi_2$, where ψ_1 and ψ_2 are two t.i. mappings from $\mathcal{A}$ to $\mathcal{P}(E)$, respectively, increasing and decreasing, the above property then holds for the dual mappings, since $\psi^* = \psi_1^* \sqcap \psi_2^*$ and ψ_1^* and ψ_2^* are, respectively, increasing and decreasing, that is,

$$\mathfrak{R}(\psi^*) = (\mathcal{K}(\psi_1^*) \times \mathcal{K}(\psi_2^*)) \cap \mathfrak{H}_{\mathcal{A}^*}.$$

Then to represent ψ, the dual form of the representation theorem may be used (see Theorem 3.2).

If ψ_1 and ψ_2 are two t.i. mappings from $\mathcal{A}$ to $\mathcal{P}(E)$, respectively, increasing and decreasing, then, from

Properties 4.3 and 4.4, the following formula can be derived

$$(\mathcal{K}(\psi_1) \times \mathcal{K}(\psi_2)) \cap \mathfrak{H}_{\mathscr{A}} =$$

$$(\mathcal{K}(\psi_1) \cap \mathcal{K}(\psi_2)) \times (\mathcal{K}(\psi_1) \cap \mathcal{K}(\psi_2)) \cap \mathfrak{H}_{\mathscr{A}}.$$

CHAPTER 5

MINIMAL REPRESENTATION THEOREMS FOR TRANSLATION INVARIANT MAPPINGS

5.1 - ALGEBRAIC ASPECTS

For the moment, let E be any non empty set.

PROPERTY 5.1 - *Let* $\mathscr{A} \subset \mathcal{P}(E)$, $\mathfrak{x}_1$ *and* $\mathfrak{x}_2$ *be two pairs in* $\mathfrak{H}_{\mathcal{P}(E)}$, $\mathcal{X}^{\mathscr{A}}_{\mathfrak{x}_1}$ *and* $\mathcal{X}^{\mathscr{A}}_{\mathfrak{x}_2}$ *be the corresponding collections, defined by* (3.3), *and let* $\mathcal{Y}^{\mathscr{A}}_{\mathfrak{x}_1}$ *and* $\mathcal{Y}^{\mathscr{A}}_{\mathfrak{x}_2}$ *be the corresponding collections, defined by* (3.11), *then*

$$\mathfrak{x}_1 \preccurlyeq \mathfrak{x}_2 \text{ implies that } \mathcal{X}^{\mathscr{A}}_{\mathfrak{x}_1} \subset \mathcal{X}^{\mathscr{A}}_{\mathfrak{x}_2} \text{ and } \mathcal{Y}^{\mathscr{A}}_{\mathfrak{x}_1} \supset \mathcal{Y}^{\mathscr{A}}_{\mathfrak{x}_2}. \qquad \square$$

PROOF: 1. For any $X \in \mathscr{A}$,

from (3.3), $\quad X \in \mathcal{X}^{\mathscr{A}}_{\mathfrak{x}_1} \Leftrightarrow (X, X) \preccurlyeq \mathfrak{x}_1$,

by assumption, $\quad \Rightarrow (X, X) \preccurlyeq \mathfrak{x}_2$,

from (3.3), $\quad \Leftrightarrow X \in \mathcal{X}^{\mathscr{A}}_{\mathfrak{x}_2}$,

consequently, $\mathcal{X}^{\mathscr{A}}_{\mathfrak{x}_1} \subset \mathcal{X}^{\mathscr{A}}_{\mathfrak{x}_2}$.

2. For any $X \in \mathscr{A}$,

from (3.11), $\quad X \in \mathcal{Y}^{\mathscr{A}}_{\mathfrak{x}_2} \Leftrightarrow (X, X) \vee \mathfrak{x}_2 \neq \mathfrak{i}$,

by assumption, $\quad \Rightarrow (X, X) \vee \mathfrak{x}_1 \neq \mathfrak{i}$,

from (3.11), $\quad \Leftrightarrow X \in \mathcal{Y}^{\mathscr{A}}_{\mathfrak{x}_1}$, consequently, $\mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}_1} \supset \mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}_2}$. □

PROPERTY 5.2 - *Let* $\mathcal{A} \subset \mathcal{P}(E)$, $\mathcal{R}^{\mathcal{A}}_{\bullet}$ *and* $\mathcal{S}^{\mathcal{A}}_{\bullet}$ *be the mappings defined, respectively, by* (3.7) *and* (3.13), *and* $\mathfrak{C}' \subset \mathfrak{C} \subset \mathfrak{H}_{\mathcal{P}(E)}$ *be such that: for any* $\mathfrak{x} \in \mathfrak{C}$ *there exists* $\mathfrak{x}' \in \mathfrak{C}'$ *such that* $\mathfrak{x} \prec \mathfrak{x}'$, *then*

$$\mathcal{R}^{\mathcal{A}}_{\mathfrak{C}} = \mathcal{R}^{\mathcal{A}}_{\mathfrak{C}'} \text{ and } \mathcal{S}^{\mathcal{A}}_{\mathfrak{C}} = \mathcal{S}^{\mathcal{A}}_{\mathfrak{C}'}. \qquad \square$$

PROOF: 1. $\mathfrak{C}' \subset \mathfrak{C}$ implies, from (3.7) and (3.13), that

$$\mathcal{R}^{\mathcal{A}}_{\mathfrak{C}'} \subset \mathcal{R}^{\mathcal{A}}_{\mathfrak{C}} \text{ and } \mathcal{S}^{\mathcal{A}}_{\mathfrak{C}} \subset \mathcal{S}^{\mathcal{A}}_{\mathfrak{C}'}.$$

2. $\mathfrak{x} \prec \mathfrak{x}'$ implies, by Property 5.1 (with $\mathfrak{x} = \mathfrak{x}_1$ and $\mathfrak{x}' = \mathfrak{x}_2$), that

$$\mathcal{X}^{\mathcal{A}}_{\mathfrak{x}} \subset \mathcal{X}^{\mathcal{A}}_{\mathfrak{x}'} \text{ and } \mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}} \supset \mathcal{Y}^{\mathcal{A}}_{\mathfrak{x}'}.$$

This leads to the two following results.

2.1 Case of $\mathcal{R}^{\mathcal{A}}_{\bullet}$: from (3.7), for every $X \in \mathcal{R}^{\mathcal{A}}_{\mathfrak{C}}$, there exists $\mathfrak{y}$, not only in $\mathfrak{C}$, but also in $\mathfrak{C}'$, such that $X \in \mathcal{X}^{\mathcal{A}}_{\mathfrak{y}}$, consequently, $X \in \mathcal{R}^{\mathcal{A}}_{\mathfrak{C}'}$ and $\mathcal{R}^{\mathcal{A}}_{\mathfrak{C}} \subset \mathcal{R}^{\mathcal{A}}_{\mathfrak{C}'}$.

2.2. Case of $\mathcal{S}^{\mathcal{A}}_{\bullet}$: from (3.13), for every $X \in \mathcal{S}^{\mathcal{A}}_{\mathfrak{C}'}$, $X \in \mathcal{Y}^{\mathcal{A}}_{\mathfrak{y}}$, not only for any $\mathfrak{y}$ in $\mathfrak{C}'$, but also in $\mathfrak{C}$, consequently, $X \in \mathcal{S}^{\mathcal{A}}_{\mathfrak{C}}$ and $\mathcal{S}^{\mathcal{A}}_{\mathfrak{C}'} \subset \mathcal{S}^{\mathcal{A}}_{\mathfrak{C}}$. □

Let $\mathcal{B} \subset \mathcal{A} \subset \mathcal{P}(E)$ and $\mathfrak{K}_{\mathcal{B}}$ be the set defined by (3.4). It is interesting to note that if $\mathfrak{x} \in \mathfrak{K}_{\mathcal{B}}$ and $\mathfrak{y} \in \mathfrak{H}_{\mathcal{A}}$ then $\mathfrak{y} \prec \mathfrak{x}$ implies that $\mathfrak{y} \in \mathfrak{K}_{\mathcal{B}}$. In other words, for any $\mathcal{B} \subset \mathcal{A}$, $\mathfrak{K}_{\mathcal{B}}$ is an ideal of $(\mathfrak{H}_{\mathcal{A}}, \prec)$. This can be proved in the following way: $\mathfrak{x} \in \mathfrak{K}_{\mathcal{B}}$ implies, from (3.4), that $\mathcal{X}_{\mathfrak{x}} \subset \mathcal{B}$.

$\mathfrak{y} \prec \mathfrak{x}$ implies, by Property 5.1, that $\mathcal{X}_{\mathfrak{y}} \subset \mathcal{X}_{\mathfrak{x}}$. Therefore, $\mathcal{X}_{\mathfrak{y}} \subset \mathcal{E}$ and, consequently, from (3.4), $\mathfrak{y} \in \mathfrak{K}_{\mathcal{E}}$.

From now on, E is the Abelian group of Chapter 2 and $\mathcal{A} \subset \mathcal{P}(E)$ is closed under translation. In order to derive a minimal representation for a t.i. mapping ψ from $\mathcal{A}$ to $\mathcal{P}(E)$, two definitions are introduced.

The first one is the definition of the basis of ψ. Let $(S, \leq)$ be a poset, m is *maximal* (respectively, *minimal*) element of $(S, \leq)$ iff $m \in S$ and for any $s \in S$, $s \geq m$ (respectively, $s \leq m$) implies that $s = m$. Let $\mathfrak{K}(\psi)$ be the set defined by (3.8) then the set $\mathfrak{B}(\psi)$ defined by

$$\mathfrak{B}(\psi) = \left\{\mathfrak{x} \in \mathfrak{H}_{\mathcal{A}} : \mathfrak{x} \text{ is maximal element of } \mathfrak{K}(\psi)\right\} \qquad (5.1)$$

is called the *basis* of ψ.

This definition of basis differs from the ones of Maragos (1985) and Dougherty and Giardina (1986) who have defined a similar notion for increasing mappings.

The second one is the definition of the so called condition of minimal representation for ψ. The subset $\mathfrak{B}$ of $\mathfrak{K}(\psi)$ is said to satisfy the *condition of minimal representation* for ψ iff for any $\mathfrak{x} \in \mathfrak{K}(\psi)$, there exists $\mathfrak{x}' \in \mathfrak{B}$ such that $\mathfrak{x} \prec \mathfrak{x}'$.

THEOREM 5.1 (Minimal representation theorem) - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation,* $\cdot \ominus \mathfrak{x}$ *be the mapping from $\mathcal{A}$ to $\mathcal{P}(E)$, defined by* (3.9), *ψ be a t.i. mapping from $\mathcal{A}$ to $\mathcal{P}(E)$, and let $\mathfrak{K}(\psi)$ and $\mathfrak{B}(\psi)$ be the sets defined, respectively, by* (3.8) *and* (5.1). *Let $\mathfrak{B}$ be any subset of $\mathfrak{K}(\psi)$ satisfying the condition of minimal representation for ψ then*

1. $\psi = \sqcup \left\{\cdot \bullet \mathfrak{x}: \mathfrak{x} \in \mathfrak{B}\right\}$;

2. *furthermore, if $\mathfrak{B}(\psi)$ is one of these $\mathfrak{B}$, i.e., if $\mathfrak{B}(\psi)$ satisfies the condition of minimal representation for ψ, then*

$$\mathfrak{B}(\psi) \subseteq \mathfrak{B}$$

and

$$\psi = \sqcup \left\{\cdot \bullet \mathfrak{x}: \mathfrak{x} \in \mathfrak{B}(\psi)\right\};$$

by definition ψ is said to have a minimal representation by a supremum. □

PROOF: 1. By Property 3.3,

$$\mathcal{K}(\psi) = \mathcal{R}_{\mathfrak{R}(\psi)},$$

by Property 5.2 (with $\mathfrak{C} = \mathfrak{R}(\psi)$ and $\mathfrak{C}' = \mathfrak{B}$),

$$= \mathcal{R}_{\mathfrak{B}},$$

from (3.7, with $\mathfrak{C} = \mathfrak{B}$) and by Property 3.5,

$$= \cup \left\{\mathcal{K}(\cdot \bullet \mathfrak{x}): \mathfrak{x} \in \mathfrak{B}\right\}.$$

Then the result of part 1 follows by Lemma 2.5.

2. $\mathfrak{B}(\psi)$ is contained in any $\mathfrak{B}$ satisfying the condition of minimal representation for ψ since, otherwise, for any $\mathfrak{x}$ in $\mathfrak{B}(\psi)$ and not in $\mathfrak{B}$ there should exist $\mathfrak{y}$ in $\mathfrak{B}$ (necessarily distinct of $\mathfrak{x}$) such that $\mathfrak{x} \leq \mathfrak{y}$, that is, $\mathfrak{B}(\psi)$ should not be the set of maximal elements of $\mathfrak{R}(\psi)$, which is a contradiction. □

The above result is important because compared to the one of Theorem 3.1, $\mathfrak{B}(\psi)$ may be much smaller than $\mathfrak{R}(\psi)$ and, consequently, it leads to an easier way to represent (or construct) the mapping ψ. Actually, such result works because of the increasing property of $\mathcal{K}(\cdot \ominus \mathfrak{X})$ with respect to $\mathfrak{X}$, that is,

$$\mathfrak{X}_1 \preceq \mathfrak{X}_2 \text{ implies that } \mathcal{K}(\cdot \ominus \mathfrak{X}_1) \subset \mathcal{K}(\cdot \ominus \mathfrak{X}_2),$$

which is equivalent to Property 5.1. The expression "minimal representation" introduced in Theorem 5.1 comes from the fact that under the condition of minimal representation $\mathfrak{B}(\psi)$ appears to be the smallest subset of $\mathfrak{K}(\psi)$ that can be used to express ψ in terms of supremum. The expression $\bigsqcup\left\{\cdot \ominus \mathfrak{X}:\ \mathfrak{X} \in \mathfrak{B}(\psi)\right\}$ in Theorem 5.1. is called the *minimal representation* for the t.i. mapping ψ *by a supremum.*

The dual form of the minimal representation by a supremum is now presented.

THEOREM 5.2 (Dual minimal representation theorem) - *Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation, $\cdot \oplus \mathfrak{X}$ be the mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ defined by (3.11), ψ be a t.i. mapping from $\mathcal{A}$ to $\mathcal{P}(E)$ and let $\mathfrak{K}(\psi^*)$ and $\mathfrak{B}(\psi^*)$ be the sets defined by (3.8) and (5.1), where ψ^* is the dual mapping of ψ. Let $\mathfrak{B}$ be any subset of $\mathfrak{K}(\psi^*)$ satisfying the condition of minimal representation for ψ^* then*

1. $\psi = \bigsqcap\left\{\cdot \oplus \mathfrak{X}:\ \mathfrak{X} \in \mathfrak{B}\right\};$

2. *furthermore, if $\mathfrak{B}(\psi^*)$ is one of these $\mathfrak{B}$, i.e., if $\mathfrak{B}(\psi^*)$ satisfies the condition of minimal representation for ψ^*, then*

$$\mathfrak{B}(\psi^*) \subset \mathfrak{B}$$

and

$$\psi = \bigsqcup\left\{\cdot \ominus \mathfrak{X}:\ \mathfrak{X} \in \mathfrak{B}(\psi^*)\right\};$$

by definition ψ is said to have a minimal representation by an infimum. $\square$

PROOF: 1. By Property 3.8,

$$\mathcal{K}(\psi) = \mathcal{P}^{\mathcal{A}}_{\mathfrak{R}(\psi^*)},$$

by Property 5.2 (with $\mathfrak{C} = \mathfrak{R}(\psi^*)$ and $\mathfrak{C}' = \mathfrak{B}$),

$$= \mathcal{P}^{\mathcal{A}}_{\mathfrak{B}},$$

from (3.13, with $\mathfrak{C} = \mathfrak{B}$) and by Property 3.7,

$$= \bigcap \left\{\mathcal{K}(\cdot \ominus \mathfrak{x}) : \mathfrak{x} \in \mathfrak{B}\right\}.$$

Then the result of part 1 follows by Lemma 2.5.

2. The arguments to prove part 2 are the same as those given to prove part 2 of Theorem 5.1. $\square$

The expression $\sqcap \left\{\cdot \ominus \mathfrak{x} : \mathfrak{x} \in \mathfrak{B}(\psi^*)\right\}$ in Theorem 5.2. is called the *minimal representation* for the t.i. mapping ψ *by an infimum.*

In what follows the special cases of increasing, decreasing and inf-separable t.i. mappings are studied.

PROPERTY 5.3 - *Let* $\mathcal{E}_1$ *and* $\mathcal{E}_2 \subset \mathcal{A} \subset \mathcal{P}(E)$, $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be, respectively, the set of minimal elements of* $\mathcal{E}_1$ *and maximal elements of* $\mathcal{E}_2$ *and* $\mathfrak{B}$ *be the set of maximal elements of* $(\mathcal{E}_1 \times \mathcal{E}_2) \cap \mathfrak{H}_{\mathcal{A}}$, *then*

$$\mathfrak{B} = (\mathcal{B}_1 \times \mathcal{B}_2) \cap \mathfrak{H}_{\mathcal{A}}.$$ $\square$

PROOF: Let $\mathfrak{x} = (A, B) \in \mathfrak{B}$, by the maximal element definition,

$$\mathfrak{x} \in (\mathcal{E}_1 \times \mathcal{E}_2) \cap \mathfrak{H}_{\mathcal{A}}$$

and

$$\left\{\mathfrak{y} \in (\mathscr{E}_1 \times \mathscr{E}_2) \cap \mathfrak{H}_{\mathscr{A}}:\ \mathfrak{x} \prec \mathfrak{y}\right\} = \langle \mathfrak{x} \rangle.$$

Since $\mathfrak{x} \in \mathfrak{H}_{\mathscr{A}}$, by the dual ideal property of $\mathfrak{H}_{\mathscr{A}}$, $\mathfrak{y} \in \mathfrak{H}_{\mathscr{A}}$ and the above equality is equivalent to:

$$\left\{\mathfrak{y} \in \mathscr{E}_1 \times \mathscr{E}_2:\ \mathfrak{x} \prec \mathfrak{y}\right\} = \langle \mathfrak{x} \rangle.$$

From (3.1), this is equivalent to $A \in \mathscr{B}_1$, $B \in \mathscr{B}_2$ and $(A, B) \in \mathfrak{H}_{\mathscr{A}}$. □

THEOREM 5.3 - *Let $\mathscr{A} \subset \mathscr{P}(E)$ be closed under translation, $\cdot \ominus \check{A}$ be the erosion by A from $\mathscr{A}$ to $\mathscr{P}(E)$, defined by (2.8), ψ be a t.i. mapping from $\mathscr{A}$ to $\mathscr{P}(E)$, $\mathscr{B}_1(\psi)$ and $\mathscr{B}_2(\psi)$ be the sets of, respectively, the minimal and the maximal elements of the kernel of ψ, $\mathscr{K}(\psi)$, defined by (2.4). If $\mathfrak{B}(\psi)$, defined by (5.1), satisfies the condition of minimal representation for ψ, then*

$$\psi = \bigsqcup \left\{\cdot \ominus \check{A}:\ A \in \mathscr{B}_1(\psi)\right\} \qquad \textit{if } \psi \textit{ is increasing,}$$

$$\psi = \bigsqcup \left\{\cdot^c \ominus \check{B}^c:\ B \in \mathscr{B}_2(\psi)\right\} \qquad \textit{if } \psi \textit{ is decreasing,}$$

$$\psi = \bigsqcup \left\{(\cdot \ominus \check{A}) \cap (\cdot^c \ominus \check{B}^c):\ A \in \mathscr{B}_1(\psi),\ B \in \mathscr{B}_2(\psi)\right\}$$

if ψ is inf-separable. □

PROOF: Let $\mathscr{B}^A(\psi)$ and $\mathscr{B}_B(\psi)$ be the collections defined by

$$\mathscr{B}^A(\psi) = \left\{X \in \mathscr{A}:\ (A, X) \in \mathfrak{B}(\psi)\right\}$$

and

$$\mathscr{B}_B(\psi) = \left\{X \in \mathscr{A}:\ (X, B) \in \mathfrak{B}(\psi)\right\},$$

for any A and $B \in \mathscr{A}$.

If $(A, B) \in \mathfrak{B}(\psi)$, then for increasing (respectively, decreasing) mapping , by Property 4.3 and from $\mathfrak{B}(\psi)$ and $\mathcal{B}(\psi)$ definitions, $A \in \mathcal{B}(\psi)$ (respectively, $B \in \mathcal{B}(\psi)$). Therefore, the result follows by applying Theorem 5.1 and if it can be proved that, for any $X \in \mathscr{A}$ and $A \in \mathcal{B}(\psi)$,

$$X \ominus \check{A} \subset \bigcup \left\{X^c \ominus \check{B}^c :\ B \in \mathcal{B}^A(\psi)\right\}$$

(respectively, for any $X \in \mathscr{A}$ and $B \in \mathcal{B}(\psi)$,

$$X^c \ominus \check{B}^c \subset \bigcup \left\{X \ominus \check{A} :\ A \in \mathcal{B}_B(\psi)\right\}).$$

1. The increasing case: let $x \in X \ominus \check{A}$ or, equivalently, $A_x \subset X$ and let $Y = X_{-x}$ then $A \subset Y$. By Property 4.2, $(A, Y) \in \mathfrak{R}(\psi)$. From the condition of minimal representation, there exists (A, Z) in $\mathfrak{B}(\psi)$ such that $(A, Z) \nmid (A, Y)$, that is, there exists $Z \in \mathcal{B}^A(\psi)$ such that $Z \supset Y$, but $Z \supset X_{-x}$ or, equivalently, $Z^c_x \subset X^c$ implies that $x \in X^c \ominus \check{Z}^c$, therefore,

$$x \in \bigcup \left\{X^c \ominus \check{B}^c :\ B \in \mathcal{B}^A(\psi)\right\}.$$

2. The decreasing case: let $x \in X^c \ominus \check{B}^c$ or, equivalently, $X \subset B_x$ and let $Y = X_{-x}$ then $Y \subset B$. By Property 4.2, $(Y, B) \in \mathfrak{R}(\psi)$. From the condition of minimal representation, there exists (Z, B) in $\mathfrak{B}(\psi)$ such that $(Z, B) \nmid (Y, B)$, that is, there exists $Z \in \mathcal{B}_B(\psi)$ such that $Z \subset Y$, but $Z \subset X_{-x}$ or, equivalently, $Z_x \subset X$ implies that $x \in X \ominus \check{Z}$, therefore,

$$x \in \bigcup \left\{X \ominus \check{A} :\ A \in \mathcal{B}_B(\psi)\right\}.$$

For inf-separable t.i. mappings, the result follows by applying Property 5.3 with $\mathcal{B}_1 = \mathcal{B}_1(\psi)$ and from (3.14) and by Theorem 5.1 and Property 4.3, since for any $X \in \mathscr{A}$ and for any (A, B) belonging to $\mathscr{B}_1(\psi) \times \mathscr{B}_2(\psi)$ but not to $\mathfrak{H}_{\mathscr{A}}$, $(X \ominus \check{A}) \cap (X^c \ominus \check{B}^c) \neq \emptyset$. □

5.2 - TOPOLOGICAL ASPECTS

For the moment, let E represent a given topological space which is assumed to be locally compact (i.e, each point in E admits a compact neighborhood), Hausdorff, and separable (i.e., the topology of E admits a countable base). Let $\mathscr{F}$ be the collection of closed subsets of E. The collection $\mathscr{F}$ is assumed to be topologized in the way proposed by Matheron (1975). Following Matheron, the selected topology on $\mathscr{F}$ is the one generated by the set of collections of the type:

$$\mathscr{F}^K = \left\{X \in \mathscr{F}:\ X \cap K = \emptyset\right\},$$

where K is a compact subset of E, and

$$\mathscr{F}_G = \left\{X \in \mathscr{F}:\ X \cap G \neq \emptyset\right\},$$

where G is an open subset of E.

In Serra (1982) and Maragos (1985) this topology is called the Hit-Miss topology.

The set of the collections of the type

$$\mathscr{F}^K, \tag{5.2a}$$

or

$$\mathscr{F}_{G_1} \cap \ldots \mathscr{F}_{G_n}, \quad (n \geq 1) \tag{5.2b}$$

or

$$\mathscr{F}^K \cap \mathscr{F}_{G_1} \cap \ldots \mathscr{F}_{G_n}, \quad (n \geq 1) \tag{5.2c}$$

is a base for the Hit-Miss topology.

The open sets in this base are collections of closed sets of E which miss a compact set of E or which hits n open sets of E or which miss a compact set of E and hits n open sets of E.

LEMMA 5.1 - *Let $\mathcal{L}$ be a subset of $\mathcal{F}$, linearly ordered (under the inclusion), then $\bigcap \mathcal{L}$ and $\overline{\bigcup \mathcal{L}}$ are adherent points of $\mathcal{L}$ in $\mathcal{F}$ (i.e., with respect to the Hit-Miss topology), that is,*

$$\bigcap \mathcal{L} \text{ and } \overline{\bigcup \mathcal{L}} \in \overline{\mathcal{L}}.$$ □

PROOF: Let $M = \bigcap \mathcal{L}$ or $\overline{\bigcup \mathcal{L}}$. It is sufficient to show that for any open set $\mathcal{A}$ of the type defined by (5.2) such that $M \in \mathcal{A}$, $\mathcal{A} \cap \mathcal{L} \neq \emptyset$. In other words, for any integer n and any $G_1, \ldots G_n$ (open sets of E), and any K (compact set of E) such that $M \cap G_i \neq \emptyset$ $(i = 1, \ldots n)$ and $M \cap K = \emptyset$, it has to be proved that there exists $X \in \mathcal{L}$ such that $X \cap G_i \neq \emptyset$ $(i = 1, \ldots n)$ and $X \cap K = \emptyset$.

1. Case of $M = \bigcap \mathcal{L}$: first, for any $X \in \mathcal{L}$, $M \subset X$, therefore, for any integer n and any open set of E, G_i $(i = 1, \ldots n)$, such that $M \cap G_i \neq \emptyset$, $X \cap G_i \neq \emptyset$, since $\emptyset \neq M \cap G_i \subset X \cap G_i$ $(i = 1, \ldots n)$; second, let K be any compact set of E such that $M \cap K = \emptyset$, that is, such that $K \subset M^c$. The set $A = M^c$ is an open set and can be written as $A = \bigcup \mathcal{M}$, where $\mathcal{M} = \left\{Y \subset E\colon\ Y^c \in \mathcal{L}\right\}$. The collection $\mathcal{M}$ is linearly ordered and is an open covering of K. The set K being a compact set of E, there exists a finite subcovering of K, say $\mathcal{M}'$. The collection $\mathcal{M}'$ being linearly ordered and finite implies that $\bigcup \mathcal{M}' \in \mathcal{M}$. Therefore, there exists $Y \in \mathcal{M}$ (namely, $Y = \bigcup \mathcal{M}'$) such that $K \subset Y \subset A$ or, equivalently, there exists $X \in \mathcal{L}$ (namely, $X = Y^c$) such that $K \cap X = \emptyset$.

2. Case of $M = \overline{\bigcup \mathscr{L}}$: first, for any $X \in \mathscr{L}$, $X \subset M$, therefore, for any compact set of E, K, such that $M \cap K = \emptyset$, $X \cap K = \emptyset$, since $X \cap K \subset M \cap K = \emptyset$; second, for any integer n and any open sets of E, G_i ($i = 1, \dots n$), such that $M \cap G_i \neq \emptyset$, by closure property, $(\bigcup \mathscr{L}) \cap G_i \neq \emptyset$. Let $x_i \in (\bigcup \mathscr{L}) \cap G_i$, by definition of $\bigcup \mathscr{L}$, there exists $X_i \in \mathscr{L}$ such that $x_i \in X_i$. In other words, there exists $X_i \in \mathscr{L}$ such that $X_i \cap G_i \neq \emptyset$. Let $\mathscr{L}'$ be the collection of the X_i ($i = 1, \dots n$). The collection $\mathscr{L}'$ being linearly ordered and finite imply that $\bigcup \mathscr{L}' \in \mathscr{L}$. Let $X = \bigcup \mathscr{L}'$, $X_i \subset X$ ($i = 1, \dots n$), which proves that there exists $X \in \mathscr{L}$ such that $X \cap G_i \neq \emptyset$ ($i = 1, \dots n$). □

LEMMA 5.2 - *Let* $\left\{A_i : i \in \mathbb{N}\right\}$ *and* $\left\{B_i : i \in \mathbb{N}\right\}$ *be two sequences in* $\mathscr{F}$ *such that* $A_i \subset B_i$ $(i \in \mathbb{N})$, $A_i \downarrow A$ *and* $B_i \uparrow B$ *in* $\mathscr{F}$, *and let* $X \in \mathscr{F}$ *such that* $A \subset X \subset B$, *then there exists a sequence* $\left\{X_i : i \in \mathbb{N}\right\}$ *in* $\mathscr{F}$ *such that* $A_i \subset X_i \subset B_i$ $(i \in \mathbb{N})$ *and* $\lim X_i = X$ *in* $\mathscr{F}$. □

PROOF: Let $X_i = (A_i \cup X) \cap B_i$ $(i \in \mathbb{N})$, then, for any $i \in \mathbb{N}$, $X_i \in \mathscr{F}$, $X_i \subset B_i$ and $A_i \subset X_i$. This last inclusion is true since,

by distributivity, $\quad X_i = (A_i \cap B_i) \cup (X \cap B_i)$,

because $A_i \subset B_i$, $\quad = A_i \cup (X \cap B_i)$.

By Corollary 3.d p. 7 in Matheron (1975) (with $F_n = X$ and $F'_n = B_n$),

$$\lim (X \cap B_i) = \overline{X \cap B} \text{ in } \mathscr{F}.$$

By Corollary 3.a p. 7 in Matheron (1975),

$$\lim A_i = A \text{ in } \mathscr{F}.$$

By Corollary 1 p. 7 in Matheron (1975), on continuity of the union, $\lim(A_i \cup (X \cap B_i)) = (\lim A_i) \cup (\lim(X \cap B_i))$ in $\mathscr{F}$. In other words, from the above three equalities on limits,

$$\lim X_i = A \cup (\overline{X \cap B}) \text{ in } \mathscr{F}.$$

By assumption, $A \subset X \subset B$ and $X \in \mathscr{F}$, therefore,

$$A \cup (\overline{X \cap B}) = A \cup \overline{X} = A \cup X = X.$$

This proves that there exists $\left\{X_i : i \in \mathbb{N}\right\}$ in $\mathscr{F}$ such that $A_i \subset X_i \subset B_i$ $(i \in \mathbb{N})$ and $\lim X_i = X$. □

Let $\mathscr{A}$ be a subcollection of $\mathscr{P}(E)$, $\mathfrak{C}$ be a subset of $\mathfrak{H}_{\mathscr{A}}$ and $\mathscr{L}_{\mathfrak{C}}$ be the subcollection of $\mathscr{A}$ defined by

$$\mathscr{L}_{\mathfrak{C}} = \left\{X \in \mathscr{A} : \exists\ (X, X') \text{ or } (X', X) \in \mathfrak{C}\right\}. \tag{5.3}$$

PROPERTY 5.4 - *Let $\mathscr{A} \subset \mathscr{P}(E)$ and $\mathfrak{C} \subset \mathfrak{H}_{\mathscr{A}}$ be linearly ordered (under $\prec$), then the subcollection $\mathscr{L}_{\mathfrak{C}}$, defined by* (5.3), *is linearly ordered (under the inclusion).* □

PROOF: For any X and $Y \in \mathscr{L}_{\mathfrak{C}}$, there exist (X, X') or $(X', X) \in \mathfrak{C}$ and (Y, Y') or $(Y', Y) \in \mathfrak{C}$.

1. If (X, X') and $(Y, Y') \in \mathfrak{C}$ or (X', X) and $(Y', Y) \in \mathfrak{C}$, then, by assumption and from (3.1), X and Y are comparable.

2. If (X, X') and $(Y', Y) \in \mathfrak{C}$ or (X', X) and $(Y, Y') \in \mathfrak{C}$, then X and Y are also comparable, since, for example, $(X, X') \prec (Y', Y)$ implies, from (3.1), that $X' \subset Y$, and, consequently, $X \subset Y$ since $X \subset X'$. □

From the above Property 5.4, if $\mathfrak{C}$ is linearly ordered, then it is always possible to choose A and B in $\mathscr{L}_{\mathfrak{C}}$ such that $A \subset B$.

PROPERTY 5.5 - *Let* $\mathscr{A} \subset \mathscr{P}(E)$, $\mathfrak{X} \subset \mathfrak{H}_{\mathscr{A}}$ *be linearly ordered (under* $\prec$*) and* $\mathscr{L}_{\mathfrak{X}}$ *be the subcollection defined by* (5.3), *then for any* A *and* B $\in \mathscr{L}_{\mathfrak{X}}$, *with* $A \subset B$, *there exists* $\mathfrak{x} \in \mathfrak{X}$ *such that* $(A, B) \prec \mathfrak{x}$. □

PROOF: If A and B $\in \mathscr{L}_{\mathfrak{X}}$, and $A \subset B$, then from (5.3), there exist $\mathfrak{x}_1$ and $\mathfrak{x}_2 \in \mathfrak{X}$ such that $\mathfrak{x}_1 = (A, A')$ or (A', A) and $\mathfrak{x}_2 = (B, B')$ or (B', B), and one of them is greater than the other. In what follows, it is proved that the greater one is always greater than (A, B).

1. If $\mathfrak{x}_1 = (A, A')$ and $\mathfrak{x}_2 = (B, B')$, then $\mathfrak{x}_2 \prec \mathfrak{x}_1$ and $(A, B) \prec (A, B') \prec (A, A') = \mathfrak{x}_1$.

2. If $\mathfrak{x}_1 = (A, A')$ and $\mathfrak{x}_2 = (B', B)$, then $\mathfrak{x}_1 \prec \mathfrak{x}_2$ or $\mathfrak{x}_2 \prec \mathfrak{x}_1$. If $\mathfrak{x}_1 \prec \mathfrak{x}_2$, $(A, B) \prec (B', B) = \mathfrak{x}_2$. If $\mathfrak{x}_2 \prec \mathfrak{x}_1$, $(A, B) \prec (A, A') = \mathfrak{x}_1$.

3. If $\mathfrak{x}_1 = (A', A)$ and $\mathfrak{x}_2 = (B', B)$, then $\mathfrak{x}_1 \prec \mathfrak{x}_2$ and $(A, B) \prec (A', B) \prec (B', B) = \mathfrak{x}_2$.

Finally, the case $\mathfrak{x}_1 = (A', A)$ and $\mathfrak{x}_2 = (B, B')$ never occurs since A is included in B. □

PROPERTY 5.6 - *Let* $\mathfrak{X} \subset \mathfrak{H}_{\mathscr{F}}$, $\mathscr{L}_{\mathfrak{X}}$ *be the subcollection defined by* (5.3, *with* $\mathscr{A} = \mathscr{F}$) *and* $\bigvee \mathfrak{X}$ *be the supremum of* $\mathfrak{X}$ *in* $\mathfrak{H}_{\mathscr{F}}$ *then*

$$\bigvee \mathfrak{X} = (\bigcap \mathscr{L}_{\mathfrak{X}}, \overline{\bigcup \mathscr{L}_{\mathfrak{X}}}) . \qquad \square$$

PROOF: Let $\mathscr{L}$ denote the collection $\mathscr{L}_{\mathfrak{X}}$. $\mathscr{L} \subset \mathscr{F}$, $\bigcap \mathscr{L}$ and $\overline{\bigcup \mathscr{L}} \in \mathscr{F}$, and $\bigcap \mathscr{L} \subset \overline{\bigcup \mathscr{L}}$, therefore

$$(\bigcap \mathscr{L}, \overline{\bigcup \mathscr{L}}) \in \mathfrak{H}_{\mathscr{F}}.$$

1. For any $(A, B) \in \mathfrak{C}$, A and $B \in \mathscr{L}$, $A \subset B$, $\bigcap \mathscr{L} \subset A \subset B \subset \bigcup \mathscr{L} \subset \overline{\bigcup \mathscr{L}}$, that is, from (3.1),

$$(A, B) \prec (\bigcap \mathscr{L}, \overline{\bigcup \mathscr{L}}).$$

This means that $(\bigcap \mathscr{L}, \overline{\bigcup \mathscr{L}})$ is an upper bound (under $\prec$) of $\mathfrak{C}$.

2. For any $(U, V) \in \mathfrak{H}_{\mathscr{F}}$,
from (5.3),

$$(A, B) \prec (U, V)\ ((A, B) \in \mathfrak{C}) \rightarrow U \subset X \subset V \quad (X \in \mathscr{L}),$$

$$\rightarrow U \subset \bigcap \mathscr{L} \text{ and } \bigcup \mathscr{L} \subset V,$$

because $V \in \mathscr{F}$, $\rightarrow U \subset \bigcap \mathscr{L}$ and $\overline{\bigcup \mathscr{L}} \subset V$,

from (3.1), $\rightarrow (\bigcap \mathscr{L}, \overline{\bigcup \mathscr{L}}) \prec (U, V)$.

This means that $(\bigcap \mathscr{L}, \overline{\bigcup \mathscr{L}})$ is the least upper bound of $\mathfrak{C}$ (under $\prec$), that is, the supremum of $\mathfrak{C}$. □

LEMMA 5.3 - *Let $\mathscr{E} \subset \mathscr{F}$ be closed in $\mathscr{F}$, $\mathfrak{R}_{\mathscr{E}}$ be the set defined by (3.4, with $\mathscr{A} = \mathscr{F}$) and $\mathfrak{C} \subset \mathfrak{R}_{\mathscr{E}}$ be linearly ordered (under $\prec$), then the supremum of $\mathfrak{C}$ in $\mathfrak{H}_{\mathscr{F}}$ is in $\mathfrak{R}_{\mathscr{E}}$, that is,*

$$\bigvee \mathfrak{C} \in \mathfrak{R}_{\mathscr{E}}.$$ □

PROOF: Let $\mathscr{L}$ denote the collection $\mathscr{L}_{\mathfrak{C}}$, defined by (5.3). By Property 5.6,

$$\bigvee \mathfrak{C} = (\bigcap \mathscr{L}, \overline{\bigcup \mathscr{L}}).$$

By applying Property 5.4, $\mathscr{L}$ is linearly ordered (under the inclusion); on the other hand, $\mathscr{L} \subset \mathscr{F}$, therefore, by Lemma 5.1, $\bigcap \mathscr{L}$ and $\overline{\bigcup \mathscr{L}} \in \overline{\mathscr{L}}$ in $\mathscr{F}$. By Theorem 1.2.1 in (Matheron, 1975), it is known that the Hit-Miss topology is separable, therefore (see for example Theorem 6.2 in (Dugundji, 1966, p. 218)) there exist two sequences $\{A_i, i \in \mathbb{N}\}$ and

$\{B_i,\ i \in \mathbb{N}\}$ in $\mathscr{L}$ such that $\lim A_i = \bigcap \mathscr{L}$ and $\lim B_i = \overline{\bigcup \mathscr{L}}$ in $\mathscr{F}$. These sequences can be chosen, respectively, decreasing and increasing and such that $A_i \subset B_i$ ($i \in \mathbb{N}$). By Corollary 3.a-b in (Matheron, 1975, p. 7),

$$\lim A_i = \bigcap \left\{A_i,\ i \in \mathbb{N}\right\}$$

and

$$\lim B_i = \overline{\bigcup \left\{B_i,\ i \in \mathbb{N}\right\}}.$$

In other words, under the linearly ordered assumption, there exist two sequences $\{A_i,\ i \in \mathbb{N}\}$ and $\{B_i,\ i \in \mathbb{N}\}$ in $\mathscr{L}$ such that $A_i \subset B_i$ ($i \in \mathbb{N}$), $A_i \downarrow \bigcap \mathscr{L}$ and $B_i \uparrow \overline{\bigcup \mathscr{L}}$. Let $X \in \mathcal{X}_{\bigvee \mathfrak{C}}$, from (3.3), $\bigcap \mathscr{L} \subset X \subset \overline{\bigcup \mathscr{L}}$ and $X \in \mathscr{F}$. By Lemma 5.2 (with $A = \bigcap \mathscr{L}$ and $B = \overline{\bigcup \mathscr{L}}$), there exists a sequence $\{X_i,\ i \in \mathbb{N}\}$ in $\mathscr{F}$ such that $A_i \subset X_i \subset B_i$ ($i \in \mathbb{N}$), that is, $X_i \in [A_i, B_i]$ ($i \in \mathbb{N}$), and $\lim X_i = X$ in $\mathscr{F}$. By Property 5.5, there exists $\mathfrak{x}_i \in \mathfrak{C}$ such that $(A_i, B_i) \dashv \mathfrak{x}_i$, that is, by Property 5.1, $[A_i, B_i] \subset \mathcal{X}_{\mathfrak{x}_i}$. In other words, for any integer i, $X_i \in \mathcal{X}_{\mathfrak{x}_i}$ with $\mathfrak{x}_i \in \mathfrak{C}$, therefore, $\mathfrak{C}$ being included in $\mathfrak{K}_{\mathscr{E}}$, from (3.4), $\mathcal{X}_{\mathfrak{x}_i} \subset \mathscr{E}$ and consequently $X_i \in \mathscr{E}$ ($i \in \mathbb{N}$). This means that X is an adherent point of $\mathscr{E}$ in $\mathscr{F}$ (see for example Theorem 6.2 in (Dugundji, 1966, p. 218)), that is, $X \in \overline{\mathscr{E}}$, but $\mathscr{E}$ has been supposed closed, therfore $X \in \mathscr{E}$ and $\mathcal{X}_{\bigvee \mathfrak{C}} \subset \mathscr{E}$. Hence, from (3.4), it has been proved that $\bigvee \mathfrak{C} \in \mathfrak{K}_{\mathscr{E}}$ in $\mathscr{F}$. □

In what follows, a sufficient condition on ψ is given under which its basis, $\mathfrak{B}(\psi)$, satisfies the condition of minimal representation for ψ.

From now on, E is the d-dimensional Euclidean space $\mathbb{R}^d$ or its subset $\mathbb{Z}^d$, equipped, respectively, with the Euclidean topology or with the relative Euclidean topology, and the t.i. mappings under consideration are from $\mathcal{F}$, the set of closed subsets of E ($\mathcal{F}$ is closed under translation), to $\mathcal{P}(E)$. It can be observed that the Euclidean topology or the relative Euclidean topology in $\mathcal{F}$ satisfy all the assumptions made on the topological space E at the begining of this section (Maragos (1985)).

Moreover, among these mappings the *upper semi-continuous* (*u.s.c*) ones from $\mathcal{F}$ to $\mathcal{F}$ are considered. A mapping ψ from $\mathcal{F}$ to $\mathcal{F}$ is u.s.c. iff for any compact subset K of E, the set $\psi^{-1}(\mathcal{F}_K)$ is closed in $\mathcal{F}$ (see Matheron (1975) p. 222).

THEOREM 5.4 (Property of the basis of an u.s.c. t.i. mapping) - *Let ψ be an u.s.c. t.i. mapping from $\mathcal{F}$ to $\mathcal{F}$ and $\mathfrak{B}(\psi)$ be the set defined by* (5.1), *then $\mathfrak{B}(\psi)$ satisfies the condition of minimal representation for ψ.* □

PROOF (The logic of this proof is the same as the one of Theorem 5.7 in Maragos (1985)): Let $\mathfrak{x} \in \mathfrak{R}(\psi)$, it is always possible to construct a subcollection of $\mathfrak{R}(\psi)$, say $\mathfrak{L}$, linearly ordered (under $\preceq$) which contains $\mathfrak{x}$, that is, $\mathfrak{x} \in \mathfrak{L} \subset \mathfrak{R}(\psi)$. By Lemma 2.1 in Maragos (1985), there exists a maximal linearly ordered (under $\preceq$) subcollection $\mathfrak{M}$ of $\mathfrak{R}(\psi)$ such that $\mathfrak{L} \subset \mathfrak{M}$. Therefore, there exists $\mathfrak{x}'$, (namely, $\mathfrak{x}' = \bigvee \mathfrak{M}$), such that, by supremum property,

$$\mathfrak{x} \preceq \bigvee \mathfrak{L} \preceq \bigvee \mathfrak{M} = \mathfrak{x}'.$$

By Proposition 8.2.1 in Matheron (1975), $\mathcal{K}(\psi)$ is closed in $\mathcal{F}$. By applying Lemma 5.3 (with $\mathcal{C} = \mathcal{K}(\psi)$ and $\mathfrak{C} = \mathfrak{M}$) and from (3.8), $\mathfrak{x}' \in \mathfrak{R}(\psi)$. Furthemore, $\mathfrak{M}$ being maximal in $\mathfrak{R}(\psi)$, $\mathfrak{x}' \in \mathfrak{B}(\psi)$, because otherwise $\mathfrak{x}'$ should not be a maximal element of $\mathfrak{R}(\psi)$ and there should exist $\mathfrak{y} \in \mathfrak{R}(\psi)$, $\mathfrak{y} \neq \mathfrak{x}'$, such that $\mathfrak{x}' \nleq \mathfrak{y}$. In other words, there should exist a subcollection of $\mathfrak{R}(\psi)$ linearly ordered bigger than $\mathfrak{M}$, (namely, $\mathfrak{M} \cup \{\mathfrak{y}\}$), and $\mathfrak{M}$ should not be maximal in $\mathfrak{R}(\psi)$ which is a contradiction. □

Theorem 5.4 is, exactly, what is needed to derive sufficient conditions to guarantee that a t.i. mapping has a minimal representation or a dual minimal representation.

THEOREM 5.5 (Minimal representation theorem - case of u.s.c. t.i. mappings) - *If ψ is an u.s.c. t.i. mapping from $\mathcal{F}$ to $\mathcal{F}$ then ψ has a minimal representation by a supremum.* □

PROOF: The result follows by applying Theorems 5.4 and 5.1 (with $\mathcal{A} = \mathcal{F}$) □

In what follows it is shown that Theorem 5.8 in Maragos (1985) (with $\mathcal{A} = \mathcal{F}$) can be derived from the above results.

COROLLARY 5.1 (Maragos (1985)) (Minimal representation of increasing t.i. u.s.c. mappings) - *Let $\cdot \ominus \check{A}$ be the erosion by A from $\mathcal{F}$ to $\mathcal{P}(E)$, defined by (2.8), ψ be an increasing u.s.c. t.i. mapping from $\mathcal{F}$ to $\mathcal{F}$ and $\mathcal{B}(\psi)$ be the set of the minimal elements of the kernel of ψ, defined by (2.4), then*

$$\psi = \bigsqcup \left\{\cdot \ominus \check{A}:\ A \in \mathcal{B}(\psi)\right\}.$$ □

PROOF: The result follows by applying Theorems 5.4 and 5.3 (with $\mathcal{A} = \mathcal{F}$). □

Let $\mathcal{G}$ be the collection of open subsets of E.

THEOREM 5.6 (Dual minimal representation theorem - case of u.s.c. t.i. mappings) - *If ψ is a t.i. mapping from $\mathcal{G}$ to $\mathcal{G}$ which has an u.s.c. dual ψ^* from $\mathcal{F}$ to $\mathcal{F}$ then ψ has a minimal representation by an infimum.* □

PROOF: If ψ^* is an u.s.c. t.i. mapping from $\mathcal{F}$ to $\mathcal{F}$ then by Theorem 5.4, $\mathfrak{B}(\psi^*)$ satisfies the condition of minimal representation for ψ^*. Hence the result follows by applying Theorem 5.2. □

When $E = \mathbb{Z}^d$ is equipped with the relative Euclidean topology, then $\mathcal{F} = \mathcal{G}$ and the above theorem even works for t.i. mappings which domain is $\mathcal{F}$, the collection of closed subsets of E.

Before ending this section, it can be observed that, for any $\mathfrak{x} \in \mathfrak{H}_{\mathcal{F}}$, the mapping $\cdot \circledast \mathfrak{x}$ from $\mathcal{F}$ to $\mathcal{P}(E)$, defined by (3.9), is u.s.c. from $\mathcal{F}$ to $\mathcal{F}$.

This can be proved in the following way: by Property 3.5, the kernel of $\cdot \circledast (A, B)$ from $\mathcal{F}$ to $\mathcal{P}(E)$ is:

$$\mathcal{K}(\cdot \circledast (A, B)) = \left\{X \in \mathcal{F}:\ A \subset X \subset B\right\}$$

$$= \left\{X \in \mathcal{F}:\ A \subset X\right\} \cap \left\{X \in \mathcal{F}:\ X \subset B\right\}.$$

By Corollary 4 p. 7 in Matheron (1975), $\left\{X \in \mathcal{F}:\ A \subset X\right\}$ and $\left\{X \in \mathcal{F}:\ X \subset B\right\}$, with $B \in \mathcal{F}$, are closed in $\mathcal{F}$, so it is for $\mathcal{K}(\cdot \circledast (A, B))$, for any $(A, B) \in \mathfrak{H}_{\mathcal{F}}$. By Proposition 8.2.1 in Matheron (1975), this is equivalent to say that $\cdot \circledast \mathfrak{x}$ for $\mathfrak{x} \in \mathfrak{H}_{\mathcal{F}}$ is an u.s.c. mapping from $\mathcal{F}$ to $\mathcal{F}$. The basis of $\cdot \circledast \mathfrak{x}$ satisfies the condition of minimal representation and is, simply, the subcollection of $\mathfrak{H}_{\mathcal{F}}$ reduced to the single pair $\mathfrak{x}$:

$$\mathfrak{B}(\cdot \circledast \mathfrak{x}) = \langle \mathfrak{x} \rangle.$$

This shows that the basis may be sometimes finite.

CHAPTER 6

EXAMPLES

In this chapter some simple examples are presented to illustrate the theory. All along this chapter E is the d-dimensional Euclidean space $\mathbb{R}^d$ or its subset $\mathbb{Z}^d$.

6.1 - COMPLEMENTARY TRANSFORMATIONS

Let $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation. The mapping $C_{\mathcal{A}}$, defined in Chapter 2, which produces the complementary set of a set in $\mathcal{A}$ is an example of t.i. mapping. Its kernel, defined by (2.4), is:

$$\mathcal{K}(C_{\mathcal{A}}) = \left\{X \in \mathcal{A}\colon\ o \notin X\right\}.$$

Since $C_{\mathcal{A}}$ is a decreasing t.i. mapping, by Property 4.3,

$$\mathfrak{K}(C_{\mathcal{A}}) = (\mathcal{A} \times \mathcal{K}(C_{\mathcal{A}})) \cap \mathfrak{H}_{\mathcal{A}}.$$

In order to say something about its basis, some assumptions on $\mathcal{A}$ must be made. If $\emptyset$ and $E - \{o\} \in \mathcal{A}$ then

$$\mathcal{K}(C_{\mathcal{A}}) = [\emptyset(E - \{o\})],$$

$(\emptyset,\ E - \{o\})$ is the greatest pair in $\mathfrak{K}(C_{\mathcal{A}})$ and the basis of $C_{\mathcal{A}}$ reduces to this single pair, that is,

$$\mathfrak{B}(C_{\mathcal{A}}) = \left\{(\emptyset,\ E - \{o\})\right\}.$$

This basis satisfies the minimal representation condition for $C_{\mathcal{A}}$, hence, by applying Theorems 5.1 and 5.3 (with $\mathcal{B}(C_{\mathcal{A}}) = \left\{E - \{o\}\right\}$), the following formulae can be, respectively, derived:

$$X^c = (X \ominus \check{\emptyset}) \cap (X^c \ominus \{\check{o}\}) \quad (X \in \mathscr{A}) \tag{6.1}$$

and

$$X^c = X^c \ominus \{\check{o}\} \quad (X \in \mathscr{A}).$$

If E is a d-dimensional Euclidean space and $\mathscr{A} = \mathscr{F}$ (the collection of closed subsets of E equipped with the Euclidean topology), then $E - \{o\}$ is an open set, that is, it does not belong to $\mathscr{A}$ and thus the above simplification does not occur. In this case, $\mathfrak{R}(C_{\mathscr{F}})$ has no maximal element. This can be seen as follows. $\mathscr{K}(C_{\mathscr{F}}) = \left\{X \in \mathscr{F}\colon\ X \subset E - \{o\}\right\}$ and for any $X \in \mathscr{K}(C_{\mathscr{F}})$ $X^c \cap E - \{o\} \neq \emptyset$ since $X^c \neq \emptyset$ $(X \neq E)$ and $X^c \neq \{o\}$ (X^c is open); hence there exists $X' \in \mathscr{K}(C_{\mathscr{F}})$ such that $X \subset X'$ and $X \neq X'$, e.g., $X' = X + \{x\}$ where $x \in X^c \cap E - \{o\}$. Since $\mathfrak{R}(C_{\mathscr{F}})$ has no maximal element, $\mathfrak{B}(C_{\mathscr{F}})$ is empty and the minimal representation condition is not fulfilled, then just Theorem 4.1 works and leads to the formula:

$$X^c = \bigcup \left\{X^c \ominus \check{B}\colon\ B \in \mathscr{G} \text{ and } o \in B\right\} \ (X \in \mathscr{F}), \tag{6.2}$$

where $\mathscr{G}$ denotes the collection of open subsets of E.

Let $\overline{C}_{\mathscr{A}}$ denote the mapping from $\mathscr{A}$ to $\mathscr{P}(E)$, defined by

$$\overline{C}_{\mathscr{A}}(X) = \overline{X^c},$$

for any $X \in \mathscr{A}$. The mapping $\overline{C}_{\mathscr{F}}$ is t.i. from $\mathscr{F}$ to $\mathscr{F}$. In this case also, $\mathfrak{R}(\overline{C}_{\mathscr{F}})$ has no maximal element. This can be seen as follows.

$$\mathscr{K}(\overline{C}_{\mathscr{F}}) = \left\{X \in \mathscr{F}\colon\ o \in \overline{X^c}\right\}$$

$$= \left\{ X \in \mathcal{F}:\ o \notin \overline{X^c}^c \right\}$$

$$= \left\{ X \in \mathcal{F}:\ o \notin \overset{\circ}{X} \right\}$$

$$= \left\{ X \in \mathcal{F}:\ \overset{\circ}{X} \subset E - \{o\} \right\}$$

and for any $X \in \mathcal{K}(\overline{C}_{\mathcal{F}})$ $X^c \cap E - \{o\} \neq \emptyset$ since $X^c \neq \emptyset$ ($X \neq E$ since $\overset{\circ}{E} = E \not\subset E - \{o\}$) and $X^c \neq \{o\}$ (X^c is open); hence there exists $X' \in \mathcal{K}(\overline{C}_{\mathcal{F}})$ such that $X \subset X'$ and $X \neq X'$, e.g., $X' = X + \{x\}$ where $x \in X^c \cap E - \{o\}$, $X' \in \mathcal{K}(\overline{C}_{\mathcal{F}})$ since $\overset{\circ}{X'} = (X \overset{\circ}{+} \{o\}) = \overset{\circ}{X} + \{o\} = \overset{\circ}{X}$. Since $\mathfrak{K}(\overline{C}_{\mathcal{F}})$ has no maximal element, $\mathfrak{B}(\overline{C}_{\mathcal{F}})$ is empty and the minimal representation condition is not fulfilled. It can be observed that Theorem 5.6 does not apply since $\overline{C}_{\mathcal{F}}$ is not u.s.c. (actually, $\overline{C}_{\mathcal{F}}$ is lower semi-continuous, see Corollary 2 p. 9 in Matheron (1975)). $\overline{C}_{\mathcal{F}}$ is decreasing and, finally, just Theorem. 4.1 works and leads to the formula:

$$\overline{X^c} = \bigcup \left\{ X^c \ominus \check{B}:\ B \in \mathcal{G} \text{ and } o \in \overline{B} \right\} \quad (X \in \mathcal{F}). \tag{6.3}$$

In formulae (6.2) and (6.3), X^c and $\check{B}$ are open sets and, consequently, $X^c \ominus \check{B}$ is a closed set. Hence, formula (6.2) shows an union of closed sets that leads to an open set and formula (6.3) shows an union of closed sets, from a bigger family, that leads to a closed set.

If $E = \mathbb{Z}^d$, equipped with the relative Euclidean topology, then $\mathcal{F} = \mathcal{P}(E)$ and if $\mathcal{A} = \mathcal{F}$, then $\mathcal{A} = \mathcal{P}(E)$, $E - \{o\} \in \mathcal{A}$, $\overline{C}_{\mathcal{F}} = C_{\mathcal{F}}$ and the above first analysis, leading to formula (6.1) holds. By Corollary 4 p. 7 in Matheron (1975),

$$\mathcal{K}(C_{\mathcal{F}}) = \left\{X \in \mathcal{F}: \; X \subset E-\{o\}\right\}$$

is a closed in $\mathcal{F}$ and, by Property 8.2.1 in Matheron (1975), $C_{\mathcal{F}}$ is u.s.c. and Theorem 5.5 can be applied, to derive formula (6.1). Actually, $C_{\mathcal{F}}$, being both lower and upper semi-continuous, is a continuous mapping with respect to the Hit-Miss topology.

6.2 - EDGE EXTRACTION

Some edge extraction mappings useful in the area of image processing may be examples of inf-separable mappings.

Let $D \in \mathcal{P}(E)$,($|D| > 1$), and $\mathcal{A} \subset \mathcal{P}(E)$ be closed under translation. The mapping ψ from $\mathcal{A}$ to $\mathcal{P}(E)$ defined by

$$\psi(X) = (X \oplus \check{D}) \cap (X^c \oplus \check{D}),$$

sometimes written,

$$= (X \oplus \check{D}) - (X \ominus \check{D}),$$

for any $X \in \mathcal{A}$, is, by Theorem 4.2, with $\psi_1 = \cdot \oplus \check{D}$ and $\psi_2 = \cdot^c \oplus \check{D}$, an inf-separable t.i. mapping. This mapping produces one version of the edge of a set in $\mathcal{A}$.

The kernels of ψ_1 and ψ_2, defined by (2.4), are:

$$\mathcal{K}(\psi_1) = \left\{X \in \mathcal{A}: \; X \cap D \neq \emptyset\right\}$$

and

$$\mathcal{K}(\psi_2) = \left\{X \in \mathcal{A}: \; X^c \cap D \neq \emptyset\right\},$$

then, by Property 4.4,

$$\mathfrak{K}(\psi) = \left\{(A, B) \in \mathfrak{H}_{\mathcal{A}}: \; A \cap D \neq \emptyset \text{ and } B^c \cap D \neq \emptyset\right\}.$$

Figure 6.1 shows a pair (A, B) in $\mathfrak{R}(\psi)$.

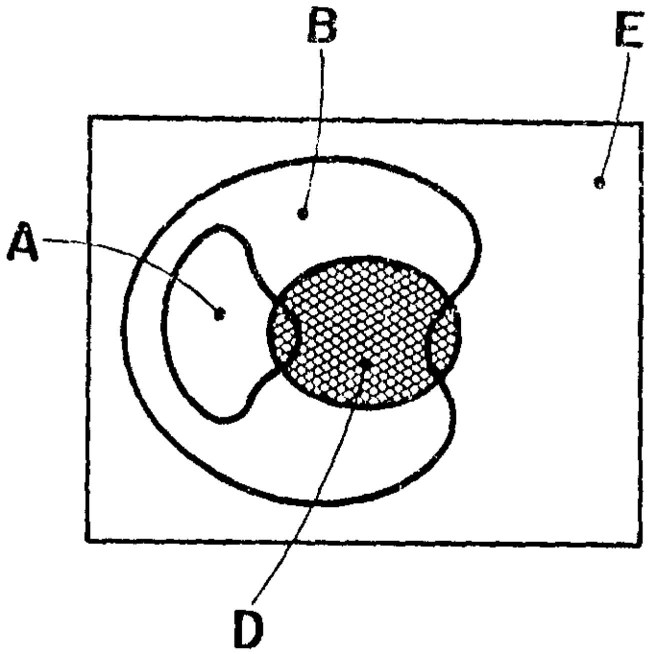

Fig. 6.1 - Example of a pair (A, B) belonging to $\mathfrak{R}((\cdot \oplus \check{D}) \cap (C \oplus \check{D}))$, the set of extremity pairs of the closed intervals contained in the kernel of an edge detection mapping caracterized by D. A and B^c must hit D and A must be contained in B.

In order to write the basis of ψ in a simple way, let us assume that $\mathscr{A} = \mathscr{P}(E)$. In this case, any subsets of the type $\{x\}$ or $\{x\}^c$ are in $\mathscr{A}$ and the sets $\mathscr{B}_1$, of the minimal elements of $\mathscr{K}(\psi_1)$, and $\mathscr{B}_2$, of the maximal elements of $\mathscr{K}(\psi_2)$, are:

$$\mathscr{B}_1 = \left\{X \in \mathscr{P}(E):\ X = \{x\} \text{ and } x \in D\right\}$$

and

$$\mathscr{B}_2 = \left\{X \in \mathscr{P}(E):\ X = \{x\}^c \text{ and } x \in D\right\}.$$

By Property 5.3,

$$\mathfrak{B}(\psi) = \left\{(A,\ B) \in \mathfrak{H}_{\mathcal{P}(E)}:\ A = \langle a \rangle,\ B = \langle b \rangle^c \text{ and } a,\ b \in D\right\}.$$

This basis satisfies the minimal representation condition for ψ, hence, by applying Theorem 5.1 and noting that $X \ominus \langle \check{h} \rangle = X_{-h}$ and $X_{-h} \cap X_{-h}^c = \emptyset$, the following formula can be derived:

$$(X \oplus \check{D}) \cap (X^c \oplus \check{D}) = \bigcup \left\{X_{-a} \cap X_{-b}^c:\ a,\ b \in D\right\}$$

$$(X \in \mathscr{A}). \qquad (6.4)$$

On the other hand, if $E = \mathbb{Z}^d$, equipped with the relative Euclidean topology, and $\mathscr{A} = \mathscr{F}$ then $\mathscr{A} = \mathcal{P}(E)$ and ψ is continuous, as intersection of two continuous mappings, that is, ψ is, in particular, u.s.c. and Theorem 5.5 can be applied to derive formula (6.4).

Of course, there are other ways to prove formula (6.4). One way is by distributivity of intersection and union and by applying Theorem 5.3 to ψ_1 and ψ_2, with $\mathscr{B}_1(\psi_1) = \mathscr{B}_1$ and $\mathscr{B}_2(\psi_2) = \mathscr{B}_2$, since, by Properties 4.3 and 5.3, with $\mathscr{A} = \mathcal{P}(E)$,

$$\mathfrak{B}(\psi_1) = \mathscr{B}_1 \times \left\{E\right\},$$

$$\mathfrak{B}(\psi_2) = \left\{\emptyset\right\} \times \mathscr{B}_2$$

and both satisfy the minimal representation condition for, respectively, ψ_1 and ψ_2. Another way is by applying Theorem 5.3 to ψ, with $\mathscr{B}_1(\psi) = \mathscr{B}_1$ and $\mathscr{B}_2(\psi) = \mathscr{B}_2$, since, by Properties 4.3 and 5.3, with $\mathscr{A} = \mathcal{P}(E)$, $\mathfrak{B}(\psi) = \mathscr{B}_1 \times \mathscr{B}_2$ and satisfies the minimal representation condition for ψ.

6.3 - REPRESENTATION FOR $\cdot \odot \mathfrak{X}$ BY AN INFIMUM

The following example shows an application of the dual minimal representation theorem.

Let $\mathscr{A} \subset \mathcal{P}(E)$, $(A, B) \in \mathfrak{S}_{\mathcal{P}(E)}$ and ψ be the mapping $\cdot \odot (A, B)$ from $\mathscr{A}$ to $\mathcal{P}(E)$, defined by (3.9). The dual mapping of ψ (see Section 3.2) is the mapping $\cdot \odot (A, B)$ from $\mathscr{A}^*$ to $\mathcal{P}(E)$, defined by (3.11). By Property 3.7,

$$\mathfrak{R}(\psi^*) = \left\{(U, V) \in \mathfrak{S}_{\mathscr{A}^*} : (U, V) \vee (A, B) \neq (\emptyset, E)\right\}.$$

Figure 6.2 shows two pairs (U, V) in $\mathfrak{R}(\psi^*)$.

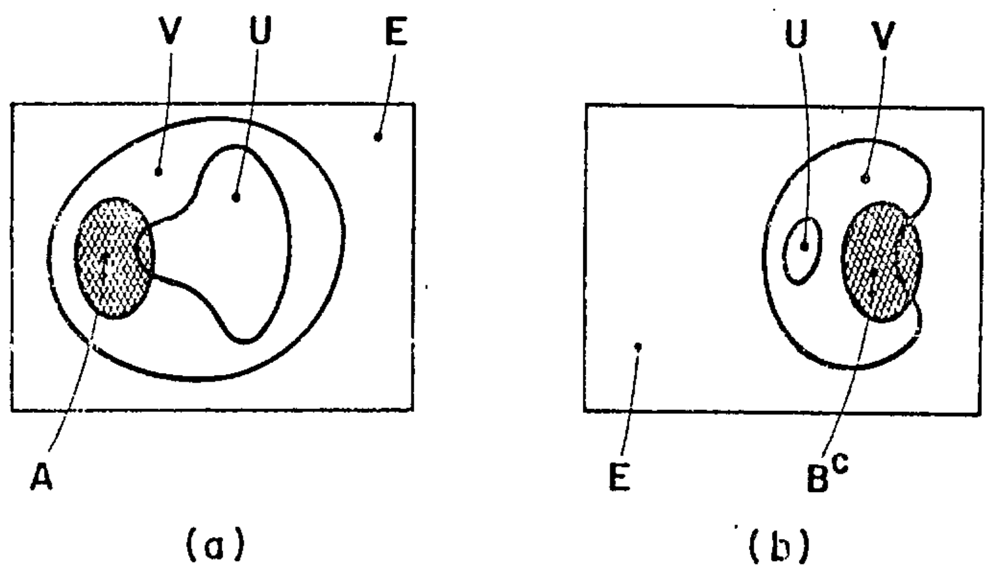

Fig. 6.2 - Example of two pairs (U, V) belonging to $\mathfrak{R}(\cdot \odot (A, B))$. U must hit A (a) or V must not contain B^c (b) and U must be contained in V.

In order to write the basis of ψ^* in a simple way, let us assume that $\mathscr{A} = \mathcal{P}(E)$. In this case any subset of E of the types $\{x\}$ or $\{x\}^c$ are in $\mathscr{A}$ and the basis of ψ^* is:

$$\mathfrak{B}(\psi^*) = \left\{(U, V) \in \mathfrak{H}_{\mathcal{P}(E)}: \begin{cases}(U, V) = (\{x\}, E) & \text{and } x \in A \\ \text{or} \\ (U, V) = (\emptyset, \{x\}^c) & \text{and } x \in B^c\end{cases}\right\}.$$

This basis satisfies the minimal representation condition for ψ^*, hence, by applying Theorem 5.2 and noting that $X \oplus \langle \check{h} \rangle = X_{-h}$, the following formula can be derived:

$$X \circledast (A, B) = (\bigcap \left\{X_{-x}: x \in A\right\}) \cap (\bigcap \left\{X^c_{-x}: x \in B^c\right\})$$

$$(X \in \mathcal{A}). \qquad (6.5)$$

On the other hand, if $E = \mathbb{Z}^d$, equipped with the relative Euclidean topology, and $\mathcal{A} = \mathcal{G}$ then $\mathcal{A} = \mathcal{F} = \mathcal{P}(E)$ and ψ^* is continuous as union of continuous mappings, that is, ψ^* is, in particular, u.s.c. and Theorem 5.6 can be applied to derive formula (6.5).

Of course, there are other ways to prove formula (6.5).

6.4 - SHAPE RECOGNITION

The last example is the so called window transformation, introduced by Crimmins and Brown (1985) in the field of automatic shape recognition.

Let $W \in \mathcal{P}(E)$, a mapping ψ from $\mathcal{A}$ to $\mathcal{P}(E)$ is called a *window transformation* with respect to a window W, if and only if, there exists a subcollection $\mathcal{D} \subset \mathcal{P}(W)$ such that

$$\psi(X) = \left\{x \in E: W \cap X_{-x} \in \mathcal{D}\right\},$$

for any $X \in \mathcal{A}$. The mapping ψ "recognizes" in particular all the shapes in $\mathcal{A}$ which are in $\mathcal{D}$ by producing a point marker. In another way,

$$\psi(X) = \left\{x \in E:\ X \in \left\{X \in \mathcal{A}:\ W \cap X \in \mathcal{D}\right\}_{x}\right\},$$

therefore, identifying with expression (2.5),

$$\mathcal{E} = \left\{X \in \mathcal{A}:\ W \cap X \in \mathcal{D}\right\},$$

and by applying Property 2.3, ψ is a t.i. mapping and its kernel is:

$$\mathcal{K}(\psi) = \left\{X \in \mathcal{A}:\ W \cap X \in \mathcal{D}\right\}.$$

Figure 6.3.a shows one typical element of $\mathcal{K}(\psi)$ when W is a rectangle and $\mathcal{D}$ contains a triangle.

Let $U \in \mathcal{D}$ and $V \in \mathcal{P}(E - W)$, and let $X = U + V$, then $W \cap X \in \mathcal{D}$. Conversely, for any X,

$$X = X \cap W + X \cap W^c,$$

thus, if $W \cap X \in \mathcal{D}$ then $X = U + V$ with $U = X \cap W \in \mathcal{D}$ and $V = X \cap W^c \in \mathcal{P}(E - W)$. Consequently,

$$\mathcal{K}(\psi) = \left\{X \in \mathcal{A}:\ X = U + V,\ U \in \mathcal{D} \text{ and } V \in \mathcal{P}(E - W)\right\}. \tag{6.6}$$

Let $U \in \mathcal{P}(W)$, if $X = U + V$ with $V \in \mathcal{P}(E - W)$ then $U \subset X \subset U + (E - W) = (W - U)^c$. Conversely, if $U \subset X \subset (W - U)^c$ then $X = U + V$ with $V \subset E - W$, that is, $V \in \mathcal{P}(E - W)$. Consequently, for any $U \in \mathcal{P}(W)$,

$$\left\{X \in \mathcal{A}:\ X = U + V \text{ and } V \in \mathcal{P}(E - W)\right\}$$

$$= \left\{X \in \mathcal{A}:\ U \subset X \subset (W - U)^c\right\}$$

and, from (6.6),

$$\mathcal{K}(\psi) = \bigcup \left\{\left\{X \in \mathcal{A}:\ U \subset X \subset (W - U)^c\right\}:\ U \in \mathcal{D}\right\}. \tag{6.7}$$

By Property 3.5 and Lemma 2.5,

$$\psi(X) = \bigcup \left\{ X \circledcirc (U, (W - U)^c) : U \in \mathcal{D} \right\} \quad (X \in \mathcal{A})$$

or equivalently, from (2.9),

$$\psi(X) = \bigcup \left\{ X \ominus (U, (W - U)) : U \in \mathcal{D} \right\} \quad (X \in \mathcal{A}). \tag{6.8}$$

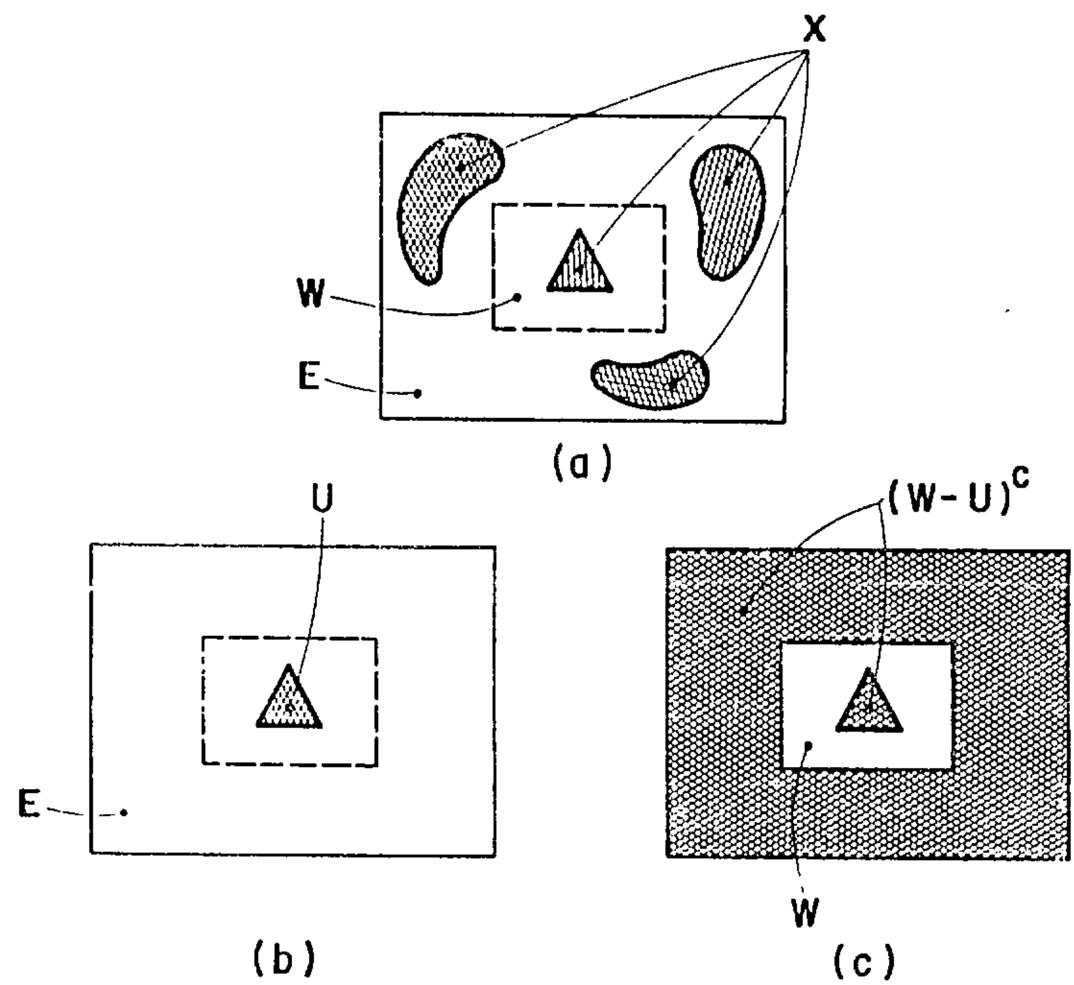

Fig. 6.3 - Example of kernel elements of a window transformation with respect to the window W and the collection $\mathcal{D}$, containing at least a triangle U. (a) shows a particular element X ($X \cap W = U$), (b) and (c) show the elements of the corresponding maximal pair $(U, (W - U)^c)$.

Formula (6.8) is the same as the one given by Maragos (1985 p. 160) and its right hand term is called here the Crimmins and Brown's representation for window transformations. In what follows, in the non trivial case for which $\mathcal{D}$ has more than one element, it is shown that, under some circumstances, such representation can be derived from the minimal representation for t.i. mappings.

For the moment, let us assume that the above collection $\mathcal{D}$ satisfies the following assumption.

ASSUMPTION 6.1 - *For any* U_1 *and* $U_2 \in \mathcal{D}$, *comparable* $(U_1 \subset U_2)$ *and distinct* $(U_1 \neq U_2)$, *there exists* $X \in \mathcal{A}$ *such that* $U_1 \subset X \cap W \subset U_2$ *and* $X \cap W \notin \mathcal{D}$. □

Under this assumption, the set $\mathfrak{K}(\psi)$, defined by (3.8), is:

$$\mathfrak{K}(\psi) = \Big\{ \mathfrak{x} \in \mathfrak{H}_{\mathcal{A}}\colon\ \mathfrak{x} = (U_1 + V_1,\ U_2 + V_2),\ U_1,\ U_2 \in \mathcal{D},$$

$$U_1 = U_2 \text{ and } \ V_1,\ V_2 \in \mathcal{P}(E - W) \Big\}.$$

This can be seen as follows. From (3.8) and (6.6), the pairs $\mathfrak{x}$ in $\mathfrak{K}(\psi)$ are of the form $\mathfrak{x} = (U_1 + V_1,\ U_2 + V_2)$ with $U_1,\ U_2 \in \mathcal{D}$ and $V_1,\ V_2 \in \mathcal{P}(E - W)$. Firstly, among such pairs those belonging to $\mathfrak{H}_{\mathcal{A}}$ and for which $U_1 = U_2 = U$ belong to $\mathfrak{K}(\psi)$ since the following statements can be successively established

$$\mathfrak{x} \prec (U,\ (W - U)^c),$$

by Property 5.1,

$$\mathcal{X}_{\mathfrak{x}} \subset \mathcal{X}_{(U,\ (W - U)^c)},$$

from (6.7),

$$\subset \mathcal{K}(\psi),$$

from (3.8),

$$\mathfrak{x} \in \mathfrak{K}(\psi).$$

Secondly, among such pairs those belonging to $\mathfrak{H}_A$ and for which $U_1 \neq U_2$ do not belong to $\mathfrak{K}(\psi)$ since, there exists, from Assumption 6.1 $X \in \mathscr{A}$ such that $X \in \mathcal{X}_{\mathfrak{x}}$ and $X \notin \mathcal{K}(\psi)$, i.e., $\mathcal{X}_{\mathfrak{x}} \not\subset \mathcal{K}(\psi)$ and, from (3.8), $\mathfrak{x} \notin \mathfrak{K}(\psi)$.

In order to write the basis of ψ in a simple way, let us assume that $(W - U)^c \in \mathscr{A}$ for any $U \in \mathscr{A} \cap \mathscr{D}$. In this case, the basis of ψ is:

$$\mathfrak{B}(\psi) = \left\{\mathfrak{x} \in \mathfrak{S}_{\mathscr{A}}: \ \mathfrak{x} = (U, \ (W - U)^c) \text{ and } U \in \mathscr{D}\right\},$$

since $(U, \ U + (E - W))$ is the maximal pair (under $\{$) of the set of pairs

$$\left\{\mathfrak{x} \in \mathfrak{S}_{\mathscr{A}}: \ \mathfrak{x} = (U + V_1, \ U + V_2) \text{ and } V_1, \ V_2 \in \mathscr{P}(E - W)\right\}.$$

Figure 6.3.b-c shows both elements of a typical pair of $\mathfrak{B}(\psi)$.

This basis satisfies the minimal representation condition for ψ, hence, by applying Theorem 5.1, the formula (6.8) can be derived.

If $\mathscr{D} \subset \mathscr{F}$, $\mathscr{A} = \mathscr{F}$ and the window W is an open subset of E, then, for any $U \in \mathscr{D}$, U and $(W - U)^c$ are closed subsets of E and, by Corollary 4 p. 7 in Matheron (1975), the sets

$$\left\{X \in \mathscr{F}: \ X \supset U\right\}$$

and

$$\left\{X \in \mathscr{F}: \ X \subset (W - U)^c\right\}$$

are closed in $\mathscr{F}$. Furthermore, if $\mathscr{D}$ is a finite collection then, from (6.7), $\mathcal{K}(\psi)$ is closed in $\mathscr{F}$ and, by Proposition 8.2.1 in Matheron (1975), this is equivalent to say that ψ is an u.s.c. mapping from $\mathscr{F}$ to $\mathscr{F}$. Hence, Theorem 5.6 can be applied to derive formula (6.8).

Actually, Assumption 6.1 was made just to derive, from the minimal representation theorem, Crimmins and Brown's representation leading to formula (6.8). If $\mathcal{D}$ does not satisfy Assumption 6.1 then, for window transformations, the minimal representation may be simpler than Crimmins and Brown's representation in the sense that ψ is the supremum of a smaller class of elementary mappings. In the increasing case, example 5.9 in Maragos (1985) illustrates this point. In the not necessarily increasing case, the K-tolerance matching is another illustrative example. Let K and W $\in \mathcal{P}(E)$, a mapping ψ from $\mathcal{A}$ to $\mathcal{P}(E)$ is called K-*tolerance matching*[1], if and only if, there exists a subcollection $\mathcal{T} \subset \mathcal{P}(W)$ such that ψ is a window transformation from $\mathcal{A}$ to $\mathcal{P}(E)$ with respect to W and the subcollection $\mathcal{D}$ defined by

$$\mathcal{D} = \left\{X \in \mathcal{P}(E):\ T \ominus \check{K} \subset X \subset (T \oplus \check{K}) \cap W \text{ and } T \in \mathcal{T}\right\}.$$

The mapping ψ "recognizes" in particular all the shapes in $\mathcal{A}$ which are similar to the ones in $\mathcal{T}$ within K-tolerant limits.

As a window transformation, ψ can be represented as in (6.8). On the other hand, by definition, $\mathcal{D}$ may not satisfy Assumption 6.1 (this depends upon $\mathcal{A}$) and a simpler representation may be suspected.

Let us assume that the above collection $\mathcal{T}$ satisfies the following assumption.

ASSUMPTION 6.2 - *For any* T_1 *and* $T_2 \in \mathcal{T}$, *comparable in the sense that* $T_1 \ominus \check{K} \subset T_2 \oplus \check{K}$ *and distinct* $(T_1 \neq T_2)$, *there exists* $X \in \mathcal{A}$ *such that* $T_1 \ominus \check{K} \subset X \cap W \subset T_2 \oplus \check{K}$ *and* $X \cap W \notin \mathcal{D}$. □

[1] *This definition has been communicated to the authors by R. M. Haralick.*

Under this assumption, the set $\mathfrak{R}(\psi)$, defined by (3.8), is:

$$\mathfrak{R}(\psi) = \Big\{\mathfrak{x} \in \mathfrak{H}_{\mathscr{A}}:\ \mathfrak{x} = (U_1 + V_1,\ U_2 + V_2),\ T \ominus \check{K} \subset U_1,$$

$$U_2 \subset (T \oplus \check{K}) \cap W,\ T \in \mathcal{T} \text{ and } V_1,\ V_2 \in \mathcal{P}(E - W)\Big\}.$$

This can be seen as follows. From (3.8) and (6.6), the pairs $\mathfrak{x}$ in $\mathfrak{R}(\psi)$ are of the form $\mathfrak{x} = (U_1 + V_1,\ U_2 + V_2)$ with $U_1,\ U_2 \in \mathcal{D}$ and $V_1,\ V_2 \in \mathcal{P}(E - W)$. Firstly, among such pairs those belonging to $\mathfrak{H}_{\mathscr{A}}$ and for which $T \ominus \check{K} \subset U_1$ and $U_2 \subset (T \oplus \check{K}) \cap W$ with $T \in \mathcal{T}$ belong to $\mathfrak{R}(\psi)$ since, for any $T \in \mathcal{T}$, the following statements can be successively established

$$\mathfrak{x} \leq (T \ominus \check{K},\ (W - (T \oplus \check{K}))^c),$$

by Property 5.1,

$$\mathcal{X}_{\mathfrak{x}} \subset \mathcal{X}_{(T \ominus \check{K},\ (W - (T \oplus \check{K}))^c)}$$

$$\subset \mathcal{K}(\psi).$$

The last inclusion is true since any $X \in \mathcal{X}_{\mathfrak{x}}$ verifies $T \ominus \check{K} \subset X \cap W \subset (T \oplus \check{K}) \cap W$, which implies, by definition of $\mathcal{D}$, that $X \cap W \in \mathcal{D}$ and consequently, by definition of ψ, that $x \in \mathcal{K}(\psi)$. Therefore, from (3.8), $\mathfrak{x} \in \mathfrak{R}(\psi)$. Secondly, among such pairs those belonging to $\mathfrak{H}_A$ and for which $T \ominus \check{K} \not\subset U_1$ or (exclusive or) $\subset U_2 \not\subset T \oplus \check{K}$, with $T \in \mathcal{T}$, i.e. (recalling that U_1 and U_2 must belong to $\mathcal{D}$), the pairs of $\mathfrak{H}_{\mathscr{A}}$ for which $T_1 \ominus \check{K} \subset U_1$, $U_2 \subset T_2 \oplus \check{K}$, with $T_1,\ T_2 \in \mathcal{T}$, $T_1 \ominus \check{K} \subset T_2 \oplus \check{K}$ and $T_1 \neq T_2$, do not belong to $\mathfrak{R}(\psi)$ since, there exists, from Assumption 6.2 $X \in \mathscr{A}$ such that $X \in \mathcal{X}_{\mathfrak{x}}$ and $X \notin \mathcal{K}(\psi)$, i.e., $\mathcal{X}_{\mathfrak{x}} \not\subset \mathcal{K}(\psi)$ and, from (3.8), $\mathfrak{x} \notin \mathfrak{R}(\psi)$.

If $T \ominus \check{K}$ and $(W - (T \oplus \check{K}))^c \in \mathscr{A}$ for any $T \in \mathcal{T}$ then the basis of ψ is precisely:

$$\mathfrak{B}(\psi) = \left\{\mathfrak{X} \in \mathfrak{H}_{\mathscr{A}}:\ \mathfrak{X} = (T \ominus \check{K},\ (W - (T \oplus \check{K}))^c) \text{ and } T \in \mathscr{T}\right\}.$$

This basis satisfies the minimal representation condition for ψ, hence by applying Theorem 5.1, the following formula can be derived:

$$\psi(X) = \bigcup \left\{X \circledast (T \ominus \check{K}, ((W - (T \oplus \check{K}))^c):\ T \in \mathscr{T}\right\}$$

$$(X \in \mathscr{A}). \qquad (6.9)$$

Formula (6.9) is simpler than formula (6.8) in the sense that $\mathscr{T} \subset \mathscr{D}$ and may be much smaller than $\mathscr{D}$.

If $\mathscr{T}$ does not satisfy Assumption 6.2 then for K-tolerance matchings, the minimal representation may even lead to a simpler formula than (6.9).

Making $K = \{o\}$, it can be observed that the K-tolerance matching with respect to $\mathscr{T}$ reduces to a window transformation with respect to $\mathscr{D} = \mathscr{T}$ and (6.9) and Assumption 6.2 reduce, respectively, to (6.8) and Assumption 6.1.

CHAPTER 7

CONCLUSION

In this paper, representations for t.i. mappings ψ are introduced. It is proved that any of these mappings can be represented as the supremum of a family of elementary mappings, $\cdot \odot \mathfrak{x}$, with $\mathfrak{x}$ in the set $\mathfrak{R}(\psi)$ of pairs of structural elements, or, in its dual form, as the infimum of another family of elementary mappings, $\cdot \odot \mathfrak{x}$, with $\mathfrak{x}$ in the set $\mathfrak{R}(\psi^*)$. For a given ψ, the simpler form, if any, may be chosen.

It is also proved that if the t.i. mapping ψ is u.s.c. then it has a minimal representation by a supremum, that is, there is a subset of $\mathfrak{R}(\psi)$, called the basis of ψ that can be used to represent ψ in a minimal way. If ψ^* is u.s.c. then ψ has a minimal representation by an infimum. It is important to note that the u.s.c. condition can be applied only for those t.i. mappings or their dual which domain is the collection of closed subsets of E, but that other t.i. mappings may have a minimal representation.

Among the examples of t.i. mappings, the interesting case of the inf-separable t.i. mappings are presented. When the t.i. mappings are only increasing their general representations reduces to Matheron's representations or Maragos' minimal representations.

Finally, three topics for future research can be outlined: the proposed representations are well adapted to be implemented on simple highly parallel architectures, which should lead to efficient image processings; in practice exact representations of the mapping ψ may not be necessary, in such case, it should be possible to construct approximations for ψ from subsets of its basis; the results derived here, for set mappings, should be extended to function mappings, offering a new tool for digital signal processing.

REFERENCES

BIRKHOFF, G. *Lattice Theory*. Providence, Rhode Island, American Mathematical Society, 1967.

CRIMMINS, T.R.; BROWN, W.M. Image algebra and automatic shape recognition. *IEEE Trans. Aerosp. and Electron. Syst.*, AES-21(1):60-69, Jan. 1985.

DOUGHERTY, E.R.; GIARDINA, C.R. A digital version of the Matheron representation theorem for increasing τ-mapping in terms of a basis for the kernel. In: IEEE COMPUTER VISION AND PATTERN RECOGNITION, Miami, June 1986. *Proceedings*. Miami, IEEE, 1986, p. 534-536.

DUGUNDJI, J. *Topology*. Boston, Allyn and Bacon, 1966.

HADWIGER, H. *Vorlesungen Uber Inhalt, Oberfläche und Isoperimetric*. Berlin, Springer-Verlag, 1957.

HARALICK, R.M.; STERNBERG, S.R.; ZHUANG, X. Image analysis using mathematical morphology. *IEEE Pattern Analysis and Machine Intelligence*, PAMI-9(4):532-550, July 1987.

GIARDINA, C.R.; DOUGHERTY, E.R. *Morphological Methods in Image and Signal Processing*. Englewood Cliffs, New Jersey, Prentice Hall, 1988.

MARAGOS, P.A. *A unified theory of translation-invariant systems with applications to morphological analysis and coding of images*. Ph.D dissertation. GA, School of Elec. Eng., Georgia Inst. Technol., July 1985.

——— A representation theory for morphological image and signal processing. *IEEE Pattern Analysis and Machine Intelligence*, PAMI-11(6):586-599, June 1989.

MATHERON, G. *Random Sets and Integral Geometry.* New York, John Wiley, 1975.

MINKOWSKI, H. Volumen und oberfläche. *Math. Annalen,* 57:447-495, 1903.

SERRA, J. *Image Analysis and Mathematical Morphology.* New York, Academic, 1982.