Expressiveness of the morphological language

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ABSTRACT

An important aspect of Mathematical Morphology is the description of set mappings by the use of a formal language, called here the Morphological Language (ML), whose vocabulary are erosions, dilations, anti-erosions, anti-dilations, infimum and supremum. Since the sixties, special machines, the Morphological maChines (MC), have been built to efficiently perform this language. These machines have proved to be very useful by solving hundreds of image analysis problems. A natural question that arises is: what class of mappings are phrases of the ML? Now, we exactly answer this question. In 1991, Banon and Barrera have proved that any translation-invariant (t.i.) mapping can be decomposed as the supremum of sup-generating mappings (the infimum of an erosion and an anti-dilation), with structuring elements that are extremities of closed set intervals contained in the kernel. Adding the hypothesis of upper semi-continuity (u.s.c.), they simplified the result by taking a minimal subcollection of sup-generating mappings. Now, we follow the same idea and generalize the concept of kernel, in order to state that any set mapping (non necessarily t.i.) can be build in a MC. W e present decompositions for set mappings in terms of sets of (non t.i.) sup-generating mappings, defined from the generalized kernels. Under the u.s.c. hypothesis, we also arrive to minimal decompositions. Some examples illustrate the main results.

1. INTRODUCTION

With the creation of the Texture Analyzer, Klein and Serra[1] initiated a new generation of image processing machines: the Morphological maChines (MC). Nowadays, a large number of these machines are available: from softwares for conventional architectures[2, 3] to implementations in silicon[4, 5] or optical technologies[6].

The family of MC has survived and proliferated so formidably thanks to its adequacy to extract image information. This capacity is largely evidenced by the solution of hundreds of image processing problems in domains so diverse as cytology[7], automation[8], cartography[9], remote sensing[10], etc.

The design of MC is supported conceptually by the theory of Mathematical Morphology. The key idea under this theory is the decomposition of mappings between complete lattices in terms of the elementary mappings of Mathematical Morphology: erosions, dilations, anti-erosions and anti-dilations[11, 12, 13, 14].

Based on this idea, we can defined a formal language[15, 16] for the description of mappings between complete lattices. The vocabulary of this language, called here the Morphological Language (ML), are erosions, dilations, anti-erosions, anti-dilation, infimum and supremum. In other words, the ML phrases describe the mappings that can be built using only elementary mappings, infimum and supremum. Under this point of view, a MC is a particular implementation of the ML.

In this paper, we present a formalization of the ML for the particular lattice $\mathcal{P}(E)$ of the parts of a non empty set E and we give some canonical expressions (phrases of the language) that can be used to decompose any set mapping. In other words, we give some results that guaranty that the ML can describe (and, of course, a MC can perform) any set mapping.

The first set mapping decomposition in terms of the elementary mappings is due to Matheron[17], who introduced, for translation-invariant (t.i) mappings, the concept of mapping kernel (a subcollection of $\mathcal{P}(E)$ that characterizes the mapping) and proved that any *increasing t.i.* set mapping can be decomposed as a supremum of erosions, with structuring elements in the mapping kernel.

Matheron's result was simplified by Maragos[18], who introduced the concept of mapping basis (a subcollection of the kernel) and proved that any upper semi-continuous (u.s.c.) increasing t.i. mapping can be represented by a supremum of erosions, with structuring elements in the mapping basis.

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The hypothesis of growth was removed by Banon and Barrera[19], who proved that any *t.i.* set mapping can be decomposed as a supremum of sup–generating mappings (infimum of an erosion and an anti–dilation), with structuring elements that are extremities of closed set intervals contained in the kernel. They also generalized the concept of mapping basis and arrived to minimal decompositions under the same u.s.c. hypothesis introduced in the increasing case.

We state, here, generalizations of the concepts of kernel and basis, in order to give the decomposition of any set mapping (non necessarily t.i.) in terms of the supremum of (non t.i.) sup–generating mappings.

In section 2, we present a grammar for the ML. In section 3, we define formally a semantics for the proposed grammar. In section 4, we give the definition and some properties of the elementary mappings. In section 5, we state the definition and some properties of sup–generating and inf–generating mappings. In section 6, we give the decomposition from the kernel and verify that this expression is a phrase of the ML. In section 7, we give decomposition from the basis for u.s.c mappings. In section 8, we give two simple examples. Finally, in section 9, we conclude and discuss some possible directions for future researches.

All the results presented in Section 5–8 are proved in Banon and Barrera[20] and Barrera[14].

2. MORPHOLOGICAL LANGUAGE GRAMMAR

In order to formally define a language, we must define a grammar (i.e., the rules that define the syntax) and a semantics (i.e., a model of interpretation for the syntax)[21, 22].

In this section, we propose a grammar for the ML. First, let state a grammar for a simpler language called Canonical Language (CL). Table 1 presents the proposed grammar for the CL using the Backus–Naur form metalanguage[15].

Table 1 - GRAMMAR OF THE CL

```
<mapping> ::= <elementary mapping> | <lattice operation>
<lattice operation> ::= <argument> <lattice operator> <argument>
<argument> ::= <elementary mapping> | (<lattice operation>)
<elementary mapping> ::= <morphological operator> <structuring function>
<structuring function> ::= <letter> | <letter> <number>
<number> ::= <digit> | <number> <digit>
<lattice operator> ::= ∨ | ∧
<morphological operator> ::= ε | δ | εᵃ | δᵃ
<letter> ::= a | b | c | d
<digit> ::= 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9
```

We denote the phrases generated by a grammar by lower case Greek letters: ϕ, ψ, etc..

The sentences that follow are some examples of the CL phrases:

$$\psi_1 ::= \epsilon_{a1} \wedge \delta^{a}{}_{b1},$$

$$\psi_2 ::= \delta_{a1} \vee \epsilon^{a}{}_{b1},$$

$$\psi_3 ::= (\epsilon_{a1} \wedge \delta^{a}{}_{b1}) \vee (\epsilon_{a2} \wedge \delta^{a}{}_{b2}),$$

$$\psi_4 ::= (\delta_{a1} \wedge \epsilon^{a}{}_{b1}) \vee (\delta_{a2} \wedge \epsilon^{a}{}_{b2}).$$

Note that the CL phrases are linked by lattice operations, in order to build new CL phrases. Figure 1 shows the syntax tree for the phrase ψ_3.

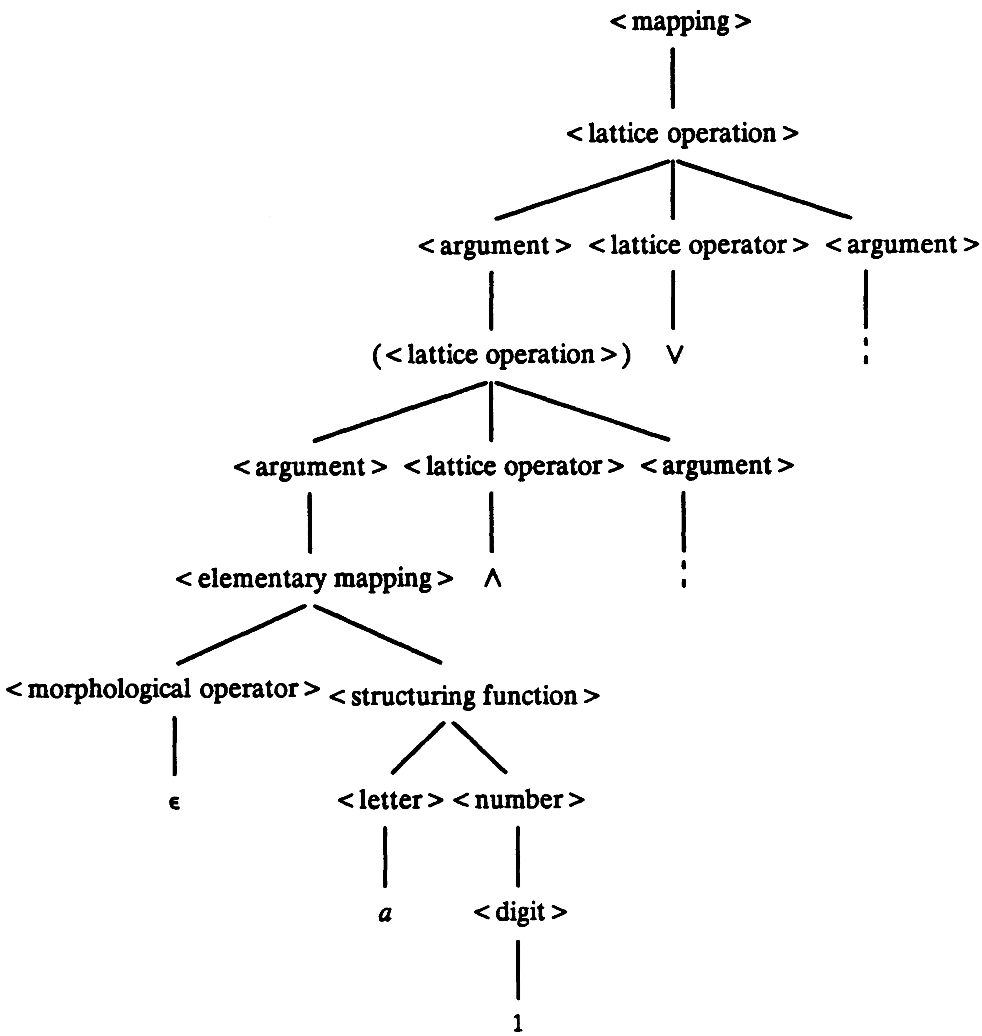

Fig. 1 – Syntax tree of the phrase ψ_3

Second, let state the ML by adding to the CL the non terminal symbol *composition*. Table 2 presents the proposed grammar for the ML.

Some examples of the ML phrases are the sentences given above (ψ_1, ψ_2, ψ_3 and ψ_4) and the ones that follow:

$$\psi_5 ::= \epsilon_a \delta_a,$$

$$\psi_6 ::= \delta_a \epsilon_a,$$

$$\psi_7 ::= (\epsilon_a \vee \epsilon_b)(\delta_a \wedge \delta_b).$$

Table 2 - GRAMMAR OF THE ML

< mapping > ::= < elementary mapping > | < lattice operation > | < composition >

< lattice operation > ::= < argument > < lattice operator > < argument >

< argument > ::= < term > | < composition >

< term > ::= < elementary mapping > | (< lattice operation >)

< composition > ::= < term > < term > | < composition > < term >

< elementary mapping > ::= < morphological operator > < structuring function >

< structuring function > ::= < letter > | < letter > < number >

< number > ::= < digit > | < number > < digit >

< lattice operator > ::= $\vee$ | $\wedge$

< morphological operator > ::= ϵ | δ | ϵ^{a} | δ^{a}

< letter > ::= *a* | *b* | *c* | *d*

< digit > ::= 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

We can observe that the ML phrases are linked by lattice operations or composition in order to build new ML phrases.

An important relationship between the CL and the ML is that the CL grammar is a subgrammar of the ML grammar, in the sense that all the phrases produced by the CL grammar can also be produced by the ML grammar. This is true, since each production rule of the CL grammar can be derived from the production rules of the ML grammar.

3. MORPHOLOGICAL LANGUAGE SEMANTICS

In order to define formally a semantics for a grammar, we must state a set of interpretation functions that map the primitive phrases into the interpretation domain[21]. The interpretation is created recursively and the execution order of the primitives in a phrase is established by the grammar.

In this section, we present formal definitions for the semantics of the CL and the ML. The phrases of these languages are interpreted as mappings between subsets of a set E.

Let $\mathcal{P}(E)$ be the collection of all subsets of E. Let $\subset$ be the usual inclusion relation between subsets. Let X^c be the complementary set of a subset X of E. We know that $(\mathcal{P}(E), \subset)$ is a complete Boolean lattice[23]. Let $\mathcal{X} \subset \mathcal{P}(E)$, then $\bigcap \mathcal{X}$, the intersection of subsets of E in $\mathcal{X}$, is the infimum of $\mathcal{X}$ in $\mathcal{P}(E)$ and $\bigcup \mathcal{X}$, the union of the subsets of E in $\mathcal{X}$, is the supremum of $\mathcal{X}$ in $\mathcal{P}(E)$. The infimum and supremum of X_1 and $X_2 \in \mathcal{P}(E)$ are, respectively, $X_1 \cap X_2$ and $X_1 \cup X_2$. Let a be a function from E to $\mathcal{P}(E)$, the function a^t, the *transpose of a*, is defined by, for any y in E,

$$a^t(y) = \{x \in E : y \in a(x)\}.$$

Let Ψ be the collection of set mappings from $\mathcal{P}(E)$ to $\mathcal{P}(E)$ (i.e., $\Psi = \mathcal{P}(E)^{\mathcal{P}(E)}$). The generic element in Ψ is denoted by the lower case Greek letter ψ. Let $\mathcal{I}$ denote the interpretation functions from subsets of the set of phrases generated by the grammar to Ψ. The Figure 2 illustrates the semantics interpretation for a phrase ψ valuated at X, that corresponds to the internal edge extraction of the shape X.

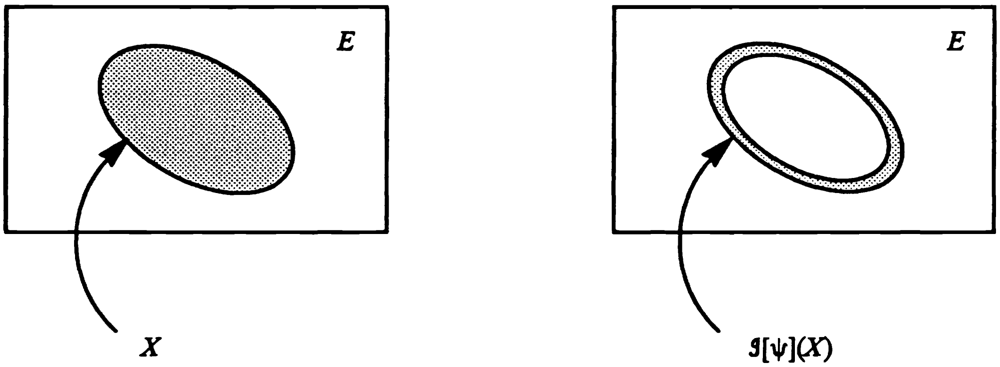

Fig. 2 – Semantics interpretation for the phrase ψ valuated at X

A formal definition of the semantics of the ML is given in Table 3 and the one of the CL is obtained just by dropping out the last sentence in that table. Since the CL grammar is a subgrammar of the ML grammar and the set of interpretation functions that defines the CL semantics is a subset of the one that defines the ML semantics, the CL is said a sublanguage of the ML. In other words, each phrase (syntax and meaning) of the CL is also a phrase of the ML.

Table 3 – SEMANTICS OF THE ML

$$\mathcal{I}[a] \equiv \mathrm{f} \in \mathcal{P}(E)^E$$

$$\mathcal{I}[\epsilon_a] \equiv \psi \in \Psi: \psi(X) = \{y \in E: \mathcal{I}[a](y) \subset X\} \quad (X \in \mathcal{P}(E))$$

$$\mathcal{I}[\delta_a] \equiv \psi \in \Psi: \psi(X) = \{y \in E: \mathcal{I}[a]^t(y) \cap X \neq \emptyset\} \quad (X \in \mathcal{P}(E))$$

$$\mathcal{I}[\epsilon^a{}_a] \equiv \psi \in \Psi: \psi(X) = \{y \in E: \mathcal{I}[a](y) \subset X\}^c \quad (X \in \mathcal{P}(E))$$

$$\mathcal{I}[\delta^a{}_a] \equiv \psi \in \Psi: \psi(X) = \{y \in E: \mathcal{I}[a]^t(y) \cap X \neq \emptyset\}^c \quad (X \in \mathcal{P}(E))$$

$$\mathcal{I}[(\phi)] \equiv \mathcal{I}[\phi]$$

$$\mathcal{I}[\phi_1 \vee \phi_2] \equiv \psi \in \Psi: \psi(X) = \mathcal{I}[\phi_1](X) \cup \mathcal{I}[\phi_2](X) \quad (X \in \mathcal{P}(E))$$

$$\mathcal{I}[\phi_1 \wedge \phi_2] \equiv \psi \in \Psi: \psi(X) = \mathcal{I}[\phi_1](X) \cap \mathcal{I}[\phi_2](X) \quad (X \in \mathcal{P}(E))$$

$$\mathcal{I}[\phi_1 \phi_2] \equiv \psi \in \Psi: \psi(X) = \mathcal{I}[\phi_1](\mathcal{I}[\phi_2](X)) \quad (X \in \mathcal{P}(E))$$

4. EROSIONS, DILATIONS, ANTI-EROSIONS AND ANTI-DILATIONS

In this section, we give the definitions of erosion, dilation, anti-erosion and anti-dilation in $(\mathcal{P}(E), \subset)$ as stated by Serra[11,12].

For any $\psi \in \Psi$, let $\psi(\mathfrak{X})$ be the image of $\mathfrak{X}$ through the mapping ψ, that is,

$$\psi(\mathfrak{X}) = \{Y \in \mathcal{P}(E) : Y = \psi(X), X \in \mathfrak{X}\}.$$

Definition 4.1 – Let $\mathfrak{X}$ be any subcollection of $\mathcal{P}(E)$. A mapping $\psi \in \Psi$ is

1) an *erosion* iff $\psi(\bigcap \mathfrak{X}) = \bigcap \psi(\mathfrak{X})$.

2) a *dilation* iff $\psi(\bigcup \mathfrak{X}) = \bigcup \psi(\mathfrak{X})$.

3) an *anti-erosion* iff $\psi(\bigcap \mathfrak{X}) = \bigcup \psi(\mathfrak{X})$.

4) an *anti-dilation* iff $\psi(\bigcup \mathfrak{X}) = \bigcap \psi(\mathfrak{X})$. □

These mappings can be parametrized by the functions from E to $\mathcal{P}(E)$, called *structuring functions*.

Proposition 4.1 (Equivalent definitions) – A mapping $\psi \in \Psi$ is

1) an erosion iff $\exists\, a \in \mathcal{P}(E)^E$: $\psi(X) = \{y \in E: a(y) \subset X\}$ $(X \in \mathcal{P}(E))$.

2) a dilation iff $\exists\, a \in \mathcal{P}(E)^E$: $\psi(X) = \{y \in E: a^t(y) \cap X \neq \emptyset\}$ $(X \in \mathcal{P}(E))$.

3) an anti-erosion iff $\exists\, a \in \mathcal{P}(E)^E$: $\psi(X) = \{y \in E: a(y) \subset X\}^c$ $(X \in \mathcal{P}(E))$.

4) an anti-dilation iff $\exists\, a \in \mathcal{P}(E)^E$: $\psi(X) = \{y \in E: a^t(y) \cap X \neq \emptyset\}^c$ $(X \in \mathcal{P}(E))$. □

Thus the phrases ϵ_a, δ_a, $\epsilon^a{}_a$ and $\delta^a{}_a$ of the ML are, respectively, the erosion, dilation, anti-erosion and anti-dilation parametrized by the structuring function a.

5. SUP-GENERATING AND INF-GENERATING MAPPINGS

In this section, we give the definitions of sup-generating and inf-generating mappings in $(\mathcal{P}(E), \subset)$ as stated by Banon and Barrera[13].

Definition 5.1 – Let $\mathfrak{X}$ be any non empty subcollection of $\mathcal{P}(E)$. A mapping $\psi \in \Psi$ is

1) a sup-generating mapping iff $\psi(\bigcap \mathfrak{X}) \cap \psi(\bigcup \mathfrak{X}) = \bigcap \psi(\mathfrak{X})$.

2) an inf-generating mapping iff $\psi(\bigcup \mathfrak{X}) \cup \psi(\bigcap \mathfrak{X}) = \bigcup \psi(\mathfrak{X})$. □

These mappings can be parametrized by a pair of structuring functions.

Proposition 5.1 (Equivalent definitions) – Let (a, b) be a pair of structuring functions. A mapping $\psi \in \Psi$ is

1) a sup-generating mapping iff $\exists\, a, b \in \mathcal{P}(E)^E$:

$$\psi(X) = \{y \in E: a(y) \subset X \subset b(y)\} \quad (X \in \mathcal{P}(E)).$$

2) an inf-generating mapping iff $\exists\, a, b \in \mathcal{P}(E)^E$:

$$\psi(X) = \{y \in E: a^t(y) \cap X \neq \emptyset \text{ or } b^t(y) \cup X \neq E\} \quad (X \in \mathcal{P}(E)).$$

□

From Proposition 5.1, we can note that a sup–generating mapping λ and an inf–generating mapping μ, characterized by a pair (a, b) of structuring functions, can be decomposed, respectively, as

$$\lambda = \epsilon_a \wedge \delta^a{}_b \quad \text{and} \quad \mu = \delta_a \vee \epsilon^a{}_b.$$

6. DECOMPOSITION OF SET MAPPINGS

In this section, we give two canonical expressions from which any set mapping can be built. Let the *kernel* $\mathcal{K}$ be the mapping from Ψ to $\mathcal{P}(\mathcal{P}(E))^E$ defined by

$$\mathcal{K}(\psi)(y) = \{X \in \mathcal{P}(E) : y \in \psi(X)\} \quad (y \in E, \psi \in \Psi).$$

Given $A \subset B$, the subset $\{X \in \mathcal{P}(E) : A \subset X \subset B\}$ is called a *closed interval* and denoted $[A, B]$ [23].

Given two structuring functions a and b from E to $\mathcal{P}(E)$, the mapping denoted $[a, b]$ and given by, for any y in E ,

$$[a, b](y) = \begin{cases} [a(y), b(y)] & \text{if } a(y) \leq b(y) \\ \emptyset & \text{otherwise} \end{cases}$$

is called an *interval function*.

Finally, let $\leq$ be the partial order relation between functions from E to $\mathcal{P}(\mathcal{P}(E))$ defined by, for any $\mathcal{K}_1$, $\mathcal{K}_2 \in \mathcal{P}(\mathcal{P}(E))^E$,

$$\mathcal{K}_1 \leq \mathcal{K}_2 \Leftrightarrow (\mathcal{K}_1(y) \subset \mathcal{K}_2(y) \quad (y \in E)).$$

Let now present the main result of this paper.

Theorem 6.1 (Decomposition by a set of sup–generating mappings) – Any set mapping ψ can be decomposed by a set of sup–generating mappings and the decomposition expression is

$$\psi = \bigvee \{\epsilon_a \wedge \delta^a{}_b : [a, b] \leq \mathcal{K}(\psi)\}.$$

□

This result works, essentially, for four reasons:

1) any function from E to $\mathcal{P}(\mathcal{P}(E))$ can exactly be written as the supremum of the family of interval functions that are less than or equal to it;

2) $\mathcal{K}(\epsilon_a \wedge \delta^a{}_b)$ is an interval function;

3) $\mathcal{K}$ is a bijection between Ψ and $\mathcal{P}(\mathcal{P}(E))^E$;

4) $\mathcal{K}$ is increasing two sides, that is, $\psi_1 \leq \psi_2 \Leftrightarrow \mathcal{K}(\psi_1) \leq \mathcal{K}(\psi_2)$.

We also have a dual result for Theorem 6.1. Let ψ^* be the dual of the mapping ψ, defined by

$$\psi^*(X) = (\psi(X^c))^c \quad (X \in \mathcal{P}(E)).$$

Any set mapping ψ can be decomposed as the infimum of inf-generating mappings and the decomposition expression is

$$\psi = \bigwedge \{\delta_a \vee \epsilon^a{}_b : [a, b] \leq \mathcal{K}(\psi^*)\}.$$

Note that both decomposition expressions are phrases of the CL, the first can be written as ψ_3 and the second as ψ_4 (see Section 2). Thus, the CL and, consequently, the ML can describe any set mapping.

7. MINIMAL DECOMPOSITION

Now, we simplify the results of Section 6, in the sense that a smaller number of sup-generating (or inf-generating) mappings is enough to perform the decompositions.

7.1. Algebraic aspects

Let $\mathcal{K}$ be any function from E to $\mathcal{P}(\mathcal{P}(E))$. An interval function less than or equal to $\mathcal{K}$ is *maximal* iff no other interval function less than or equal to $\mathcal{K}$ is greater than it.

The set $\mathbf{B}(\psi)$ of all the maximal interval functions less than or equal to $\mathcal{K}(\psi)$ is called the *basis of* ψ.

The set $\mathbf{B}$ of interval functions less than or equal to $\mathcal{K}(\psi)$ is said to satisfy the *decomposition condition for* ψ iff for any interval function less than or equal to $\mathcal{K}(\psi)$ there exists an interval function in $\mathbf{B}$ that is greater than it.

Theorem 7.1 (Decomposition by a least set of sup-generating mappings) – Let ψ be any set mapping and let $\mathbf{B}$ be a set of interval functions less than or equal to $\mathcal{K}(\psi)$, satisfying the decomposition condition for ψ, then

$$\psi = \bigvee \{\epsilon_a \wedge \delta^a{}_b : [a, b] \in \mathbf{B}\}.$$

Furthermore, if $\mathbf{B}(\psi)$, its basis, satisfies the decomposition condition for ψ, then

$$\mathbf{B}(\psi) \subset \mathbf{B},$$

$$\psi = \bigvee \{\epsilon_a \wedge \delta^a{}_b : [\mathrm{a}, \mathrm{b}] \in \mathbf{B}(\psi)\}$$

and ψ is said to have a minimal decomposition by a set of sup-generating mappings. □

Theorem 7.1 also has a dual result. Let ψ be any set mapping and let $\mathbf{B}$ be a set of interval functions less than or equal to $\mathcal{K}(\psi^*)$, satisfying the decomposition condition for ψ^*, then

$$\psi = \bigwedge \{\delta_a \vee \epsilon^a{}_b : [a, b] \in \mathbf{B}\}.$$

Furthermore, if $\mathbf{B}(\psi^*)$, the basis of its dual, satisfies the decomposition condition for ψ^*, then

$$\mathbf{B}(\psi^*) \subset \mathbf{B},$$

$$\psi = \bigwedge \{\delta_a \vee \epsilon^a{}_b : [a, b] \in \mathbf{B}(\psi^*)\},$$

and ψ is said to have a minimal decomposition by a set of inf-generating mappings.

We should note that when E is finite, $\mathbf{B}(\psi)$ and $\mathbf{B}(\psi^*)$ always satisfy the decomposition condition for, respectively, ψ and ψ^*.

7.2. Topological aspects

Now, we give a sufficient condition for which the decomposition condition is satisfied.

Throughout this subsection E is locally compact (i.e., each point in E admits a compact neighborhood), Hausdorff and separable (i.e., the topology of E admits a countable base) topological space.

In order to describe the sufficient condition, we use the Hit–Miss topology on the collection $\mathcal{F}$ of closed subsets of E. The Hit–Miss topology on $\mathcal{F}$ is generated by the set of collections of the type

$$\mathcal{F}^K = \{X \in \mathcal{F}: X \cap K = \emptyset\},$$

where K is a compact subset of E, and

$$\mathcal{F}_G = \{X \in \mathcal{F}: X \cap G \neq \emptyset\},$$

where G is an open subset of E[17, 24].

A mapping ψ from $\mathcal{F}$ to $\mathcal{F}$ is upper semi–continuous (u.s.c.) iff, for any compact subset K of E, the set $\psi^{-1}(\mathcal{F}^K)$ is open in $\mathcal{F}$ (see Matheron[17], p.8).

Theorem 7.2 – Let ψ be an u.s.c mapping from $\mathcal{F}$ to $\mathcal{F}$, then its basis $\mathbf{B}(\psi)$ satisfies the decomposition condition for ψ. □

Let $\mathcal{G}$ be the collection of open subsets of E. Theorem 7.2 has the following immediate consequences:

1) if ψ is an u.s.c. mapping from $\mathcal{F}$ to $\mathcal{F}$, then ψ has a minimal decomposition by a set of sup–generating mappings;

2) if ψ is a mapping from $\mathcal{G}$ to $\mathcal{G}$ that has an u.s.c. dual ψ^*, then ψ has a minimal decomposition by a set of inf–generating mappings.

8. EXAMPLES

Now, we illustrate the concept of basis by analyzing two simple examples for which the set E is assumed finite. As a first example, let $\lambda_{a,b}$ be the sup–generating mapping

$$\lambda_{a,b}(X) = \{y \in E: a(y) \subset X \subset b(y)\} \quad (X \in \mathcal{P}(E)).$$

The kernel and the basis of $\lambda_{a,b}$ are given, respectively, by

$$\mathcal{K}(\lambda_{a,b}) = [a, b]. \quad \text{and} \quad \mathbf{B}(\lambda_{a,b}) = \{[a, b]\}.$$

Figure 3 shows a set X that belongs to $\mathcal{K}(\lambda_{a,b})(y)$.

As a second example[20], let γ_a be an *opening by the structuring function a*, that is,

$$\gamma_a(X) = \delta_a \varepsilon_a(X) \quad (X \in \mathcal{P}(E)).$$

The kernel of γ_a is given by, for any $y \in E$,

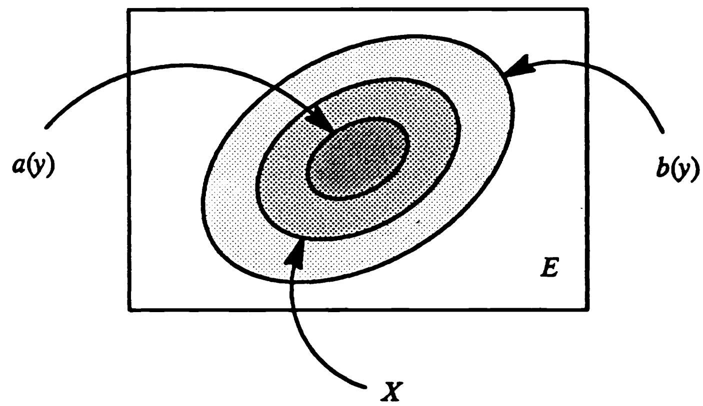

Fig. 3 – An element X of the kernel of a sup–generating mapping, with structuring functions a and b, valuated at y

$$\mathcal{K}(\gamma_a)(y) = \bigcup \{\mathcal{K}(\epsilon_a)(x) : x \in a^t(y)\},$$

where,

$$\mathcal{K}(\epsilon_a)(y) = \{X \in \mathcal{P}(E) : a(y) \subset X\}.$$

The following set **B** of interval functions less than or equal to $\mathcal{K}(\gamma_a)$ satisfies the decomposition condition for γ_a

$$\mathbf{B} = \{[u, v] : \forall y \in E, y \in u(y) \in a(E) \text{ and } v(y) = E\},$$

where, $a(E)$ is the image of E through the structuring function a.

The basis of γ_a is given by

$$\mathbf{B}(\psi) = \{[u, v] : \forall y \in E, u(y) \text{ is a minimal element of } \{a(x) : x \in a^t(y)\} \text{ and } v(y) = E\} \subset \mathbf{B}.$$

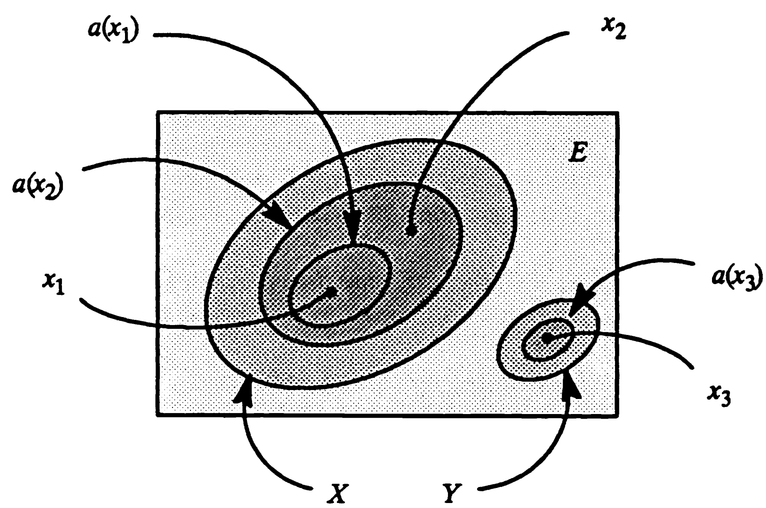

Fig. 4 – Two elements X and Y of the kernel of an opening by the structuring function a, valuated at y

Figure 4 shows two sets X and Y that belong to $\mathcal{K}(\gamma_a)(y)$ with $a^t(y) = \{x_1, x_2, x_3\}$.

9. CONCLUSION

Motivated by the enormous potentialities of the MC, verified in practice, we proposed a formal definition for the ML and studied its expressiveness, by presenting some canonical expressions to represent set mappings.

We stated formally the notion of ML by defining a grammar (set of rules that describes the syntax) and a semantics (model of interpretation for the syntax). The notion of MC was conceptualized as a physical realization of the ML.

We generalize the notion of kernel and, based on it, we presented two canonical expressions, that are phrases of the ML, for representing any set mapping. In other words, we verified that the ML can describe (and, of course, a MC can perform) any set mapping.

We generalize the notion of basis, taking the maximal interval functions of the mapping kernel, in order to simplify the decomposition expressions. In the finite case (E finite), any set mapping has a minimal decomposition. In the general case, we verified that the upper semi–continuity is a sufficient condition to guaranty the minimal decomposition.

The decomposition expressions presented are strongly parallel, since they are built only by elementary mappings linked by infimum and supremum. This characteristic may lead to prohibitive implementations, since the decompositions may use an enormous number of elementary mappings.

In the ML, besides infimum and supremum, the elementary mappings can be linked by composition. This property is not used in the presented canonical decompositions and it may be a way for getting simpler ones. For example, the t.i. opening can be decomposed through the composition of two elementary mappings (an erosion and a dilation), but its decomposition as a supremum of erosions may use a lot of elementary mappings.

Finally, the CL is enough to express any set mapping, but we may say that it is less expressive than the ML, since the representations in the latter may involve a smaller number of elementary mappings.

10. ACKNOWLEDGMENTS

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