On-Off Collective Imperfect Phase Synchronization and Bursts in Wave Energy in a Turbulent State

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A new type of synchronization, on-off collective imperfect phase synchronization, is found in a turbulent state. In the driver frame the nonlinear wave system can be transformed to a set of coupled oscillators moving in a potential related to the unstable steady wave. In "on" stages the oscillators in different spatial scales adjust themselves to collective imperfect phase synchronization, inducing strong bursts in the wave energy. The interspike intervals display a power-law distribution. In addition to the embedded saddle point, it is emphasized that the delocalization of the master mode also plays an important role in developing the on-off synchronization.

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I. Introduction.—Turbulence, or more generally speaking, spatiotemporal chaos (STC) is a very important topic in fluids, plasmas, nonlinear optics, and other fields [1]. A fully developed turbulence has little correlation in space. However, in Fourier space it has a power law [2], indicating strong interaction among different scales. This is in contrast to a spatially regular (SR) state where most smaller scales are quiescent [3]. On the other hand, a turbulent system often shows strong integral dynamical events. For instance, bursts of solar flares are considered to be related to plasma turbulence of the nonlinear solar dynamics [4]. In brain dynamics, it is believed that phase synchronization of ongoing rhythms across different spatiotemporal scales is necessary for higher cognitive tasks [5]. Is such an event the result of collective motion of a large variety of scales? If it is, how do they cooperate with each other to bring it about?

In recent years synchronization of coupled chaotic oscillators has attracted great attention [6], which provides a clue to answer the above questions. Several types of synchronization features have been revealed [7]. Synchronization between oscillators can be perfect or imperfect; e.g., it is found that when the intrinsic time scale is not bounded from above due to an embedded saddle point, coupled oscillators in the Lorenz system can adjust to imperfect synchronization [8].

In the present Letter we show that in a type of STC an on-off collective imperfect synchronization (CIS) can be developed among different spatial scales; in on stages the mode energies are effectively superimposed giving rise to integral energy bursts. In Sec. II the model equation and a transition from the SR to STC are briefly introduced. In Sec. III the on-off CIS in the STC is demonstrated. In Sec. IV the physical mechanism of CIS is discussed; in addition to the saddle point, the effect of delocalization of the master mode is also addressed.

II. Spatially coherent and spatiotemporal chaotic states.—In Ref. [9] a crisis-induced transition from the SR to STC is found in the nonlinear wave equation:

$$\frac{\partial\phi}{\partial t} + a\frac{\partial^3\phi}{\partial t\partial x^2} + c\frac{\partial\phi}{\partial x} + f\phi\frac{\partial\phi}{\partial x} = -\gamma\phi - \varepsilon\sin(x - \Omega t). \qquad (1)$$

A periodic boundary condition of 2π length is applied. $a < 0$, c, f, and γ are fixed parameters. When $\Omega = \epsilon = 0$ the above equation can be derived from magnetized plasmas to describe the drift wave or from fluids [10]. For Ω in certain regimes, a transition from the SR to STC occurs at a critical $\epsilon = \epsilon_c$. When $\Omega = 0.65$, $\epsilon_c \approx 0.20$. In the following we use $\epsilon = 0.22$ for the STC. The SR and STC show spatial spectra of exponential and power law, respectively.

Figure 1 shows the "averaged wave number" [11,12] $\sqrt{N^2} = \sqrt{\sum_{n=1}^{N} n^2|\phi_n|^2/\sum_{n=1}^{N}|\phi_n|^2}$, for the SR and STC, respectively; here ϕ_n is the n'th Fourier component of $\phi(x)$. $N = 512$ is used in the numerical calculation based on the pseudospectral method. In the plot one can see that only a few large-scale modes are active in the SR; in contrast, in the STC the saturation level is much higher, as time moves on the curve becomes more and more fluctuating, indicating that the energy may, in some way, transfer across very different scales.

On the other hand, although in the STC the spatial behavior seems irregular, it is by no means entirely incoherent; in fact, coherent structures of different scales are observed to emerge and annihilate intermittently. Such a "soliton gas" is typical of turbulence in many systems, which suggests that correlation among the modes—perhaps temporarily—can happen even in strong turbulence. To understand all these phenomena, it is essential to investigate the motion in different scales and their relations.

III. On-off imperfect phase synchronization and bursts in wave energy.—To study the behaviors of different scales it is helpful to shift the system (1) to a driver frame with $\xi = x - \Omega t, \tau = t$, in which a steady wave (SW) $\phi_0(\xi)$ satisfies $\partial\phi_0/\partial\tau = 0$. For given parameters, by expanding $\phi_0(\xi) = \sum_{k=1}^{N} \phi_{0,k} \equiv \sum_{k=1}^{N} A_k \cos(k\xi + \theta_k)$, $\{A_k, \theta_k\}$ can be solved. It is easy to see that an SW $\phi_0(\xi)$ is a fixed point in the Fourier space. Then by setting $\phi(\xi, \tau) = \phi_0(\xi) + \delta\phi(\xi, \tau)$, the perturbation wave (PW) $\delta\phi(\xi, \tau)$ is governed by

$$\frac{\partial}{\partial\tau}\left[1 + a\frac{\partial^2}{\partial\xi^2}\right]\delta\phi - \Omega\frac{\partial}{\partial\xi}\left[1 + a\frac{\partial^2}{\partial\xi^2}\right]\delta\phi + c\frac{\partial}{\partial\xi}\delta\phi + \gamma\delta\phi + f\frac{\partial}{\partial\xi}[\phi_0(\xi)\delta\phi] + f\delta\phi\frac{\partial}{\partial\xi}\delta\phi = 0. \quad (2)$$

Expanding $\delta\phi(\xi, \tau) = \sum_k \delta\phi_k \equiv \sum_k b_k(\tau)\cos[k\xi + \alpha(\tau)]$, we get

$$\begin{aligned}\frac{db_k}{d\tau} = -\frac{\gamma}{1 - ak^2}b_k + \frac{fk}{2(1 - ak^2)}\Bigg\{&\sum_{i+j=k}[A_ib_j\sin(\theta_i + \alpha_j - \alpha_k) + b_ib_j\sin(\alpha_i + \alpha_j - \alpha_k)/2]\\ &+ \sum_{i-j=k}[A_ib_j\sin(\theta_i - \alpha_j - \alpha_k) + b_ib_j\sin(\alpha_i - \alpha_j - \alpha_k)/2]\\ &+ \sum_{j-i=k}[A_ib_j\sin(-\theta_i + \alpha_j - \alpha_k) + b_ib_j\sin(-\alpha_i + \alpha_j - \alpha_k)/2]\Bigg\},\\ \frac{d\alpha_k}{d\tau} = -k\left[\frac{c}{1 - ak^2} - \Omega\right] - \frac{fk}{2(1 - ak^2)b_k}\Bigg\{&\sum_{i+j=k}[A_ib_j\cos(\theta_i + \alpha_j - \alpha_k) + b_ib_j\cos(\alpha_i + \alpha_j - \alpha_k)/2]\\ &+ \sum_{i-j=k}[A_ib_j\cos(\theta_i - \alpha_j - \alpha_k) + b_ib_j\cos(\alpha_i - \alpha_j - \alpha_k)/2]\\ &+ \sum_{j-i=k}[A_ib_j\cos(-\theta_i + \alpha_j - \alpha_k) + b_ib_j\cos(-\alpha_i + \alpha_j - \alpha_k)/2]\Bigg\}\\ &(k = 1, 2, \cdots, N \to \infty). \quad (3)\end{aligned}$$

After obtaining $\{A_k, \theta_k\}$, $\{b_k(\tau), \alpha_k(\tau)\}$ can then be solved. $\{b_k(\tau), \alpha_k(\tau)\}$ is a set of coupled oscillators; their motions are influenced also by $\phi_0(\xi)$ as if the latter is a potential [see also Eq. (2)]. Since $\phi_0(\xi)$ has the structure of a solitary wave, the characteristic phenomena of the STC, e.g., its power-law spectrum, are actually displayed in the motion of $\delta\phi(\xi, \tau)$.

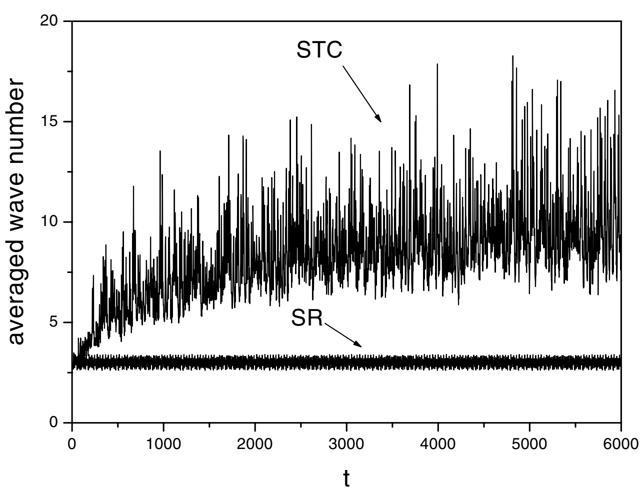

FIG. 1. Averaged wave number $\sqrt{N^2}$ for STC wave ($\Omega = 0.65, \epsilon = 0.22$) and SR wave ($\Omega = 0.65, \epsilon = 0.19$).

First let us analyze the behavior of "wave energy" defined as $E(t) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2}[\phi^2 - a\phi_x^2]dx$. It is positive for $a < 0$ as required by the physics. For the STC Fig. 2(a) shows $\delta E(\tau) \equiv E(\tau) - E_0 = \sum_k \delta E_k \equiv \sum_k (1 - ak^2)[A_kb_k\cos(\theta_k - \alpha_k)/2 + b_k^2/4]$; here $E_0 \equiv E(\phi_0)$ is a constant. In its very erratic curve one can identify many pronounced bursts; between two strong bursts the motion is fluctuating in relatively lower levels. It is worth pointing out that this type of behavior is very common in fully developed turbulence [13].

Let us turn to Eq. (3) to study the origin of the bursts. If all the mode couplings are discarded, a given oscillator $\delta\phi_k(\xi, \tau)$ has a frequency $-k[c/(1 - ak^2) - \Omega]$. Because of coupling to SW modes $\{A_k, \theta_k\}$, the eigenfrequencies of $\delta\phi_k$ are altered nonlinearly. It has been shown that in the present case one of the nonlinear eigenfrequencies vanishes; that is, the corresponding SW $\phi_0(\xi)$ is unstable due to a saddle instability [9], which is a saddle point embedded in the STC attractor. In the following we can see that this set of oscillators can adjust to a special type of synchronization: it is imperfect due to the saddle point, and it is in an on-off manner.

Figures 2(b)–2(d) show the phase difference $\Delta\alpha_{1k} \equiv \alpha_1(\tau) - \alpha_k(\tau)$ for $k = 2, 3, 4$, respectively. By comparing Figs. 2(b)–2(d) with Fig. 2(a) an interesting phenomenon

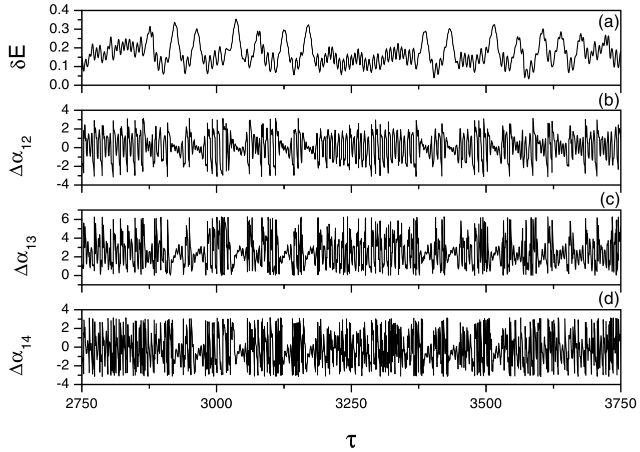

FIG. 2. (a) $\delta E(\tau)$, (b) $\Delta\alpha_{12}(\tau)$, (c) $\Delta\alpha_{13}$, (d) $\Delta\alpha_{14}$. In (b)–(d) mod(2π) is taken.

is, right at the positions where $\delta E(\tau)$ shows strong bursts, $\Delta\alpha_{1k}$ has very little variations, which indicates that they are nearly synchronized; whereas between two bursts, $\Delta\alpha_{1k}$ often slips over 2π [in the plots mod(2π) have been taken], the phases are no longer locked. That is, the behavior of the total wave energy is closely related to the "on-off" synchronization among the modes.

Figure 3(a) shows $\delta E(\tau)$, 3(b) $\alpha_k(\tau)$ for $k = 1, 2$, and 3(c) b_k for $k = 1$–3, respectively. One can see that at the strong bursts of $\delta E(\tau)$, $\alpha_{k=1}$ and $\alpha_{k=2}$ are synchronized imperfectly, and the mode amplitudes are also nearly synchronized, at least for small k.

Since in the CIS the PW phases $\{\alpha_k\}$ are ~ 0 or shift to $\sim\pi$, one can define a collective correlation function,

$$\langle C_\alpha^N(\tau)\rangle = \langle|\Pi_{k=1}^N \cos\alpha_k(\tau)|\rangle. \tag{4}$$

Figure 4(a) shows $\delta E(\tau)$ and 4(b) $\langle C_\alpha^4(\tau)\rangle$. The average $\langle\,\rangle$ is taken in a driving period. The correspondence between the sharp spikes in $\langle C_\alpha^4\rangle$ and the bursts in δE is very clear.

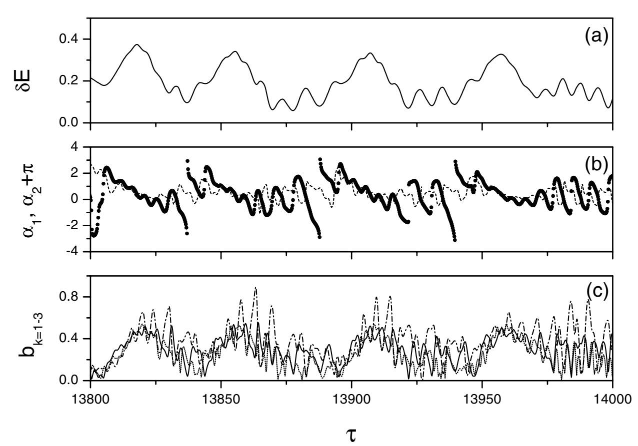

FIG. 3. (a) $\delta E(\tau)$, (b) $\alpha_k(\tau)$ for $k = 1$ (bullets) and $k = 2$ (dashed curve), mod(2π) is taken, (c) b_k for $k = 1$ (solid curve), $k = 2$ (dot-dashed curve), and $k = 3$ (dotted curve).

In contrast, for the corresponding lower-level fluctuations in $\delta E(t)$, the correlation is weak.

From the above results one can see that even a turbulent system can develop a certain type of synchronization. It is significant that in the "on" stages the system has an integral manifestation in the sense that it can give rise to total energy bursts and the different scales are more tightly correlated. We would expect that as time moves on, smaller and smaller scales join the CIS, inducing energy cascading to smaller scales.

As mentioned above, the motion of oscillators is influenced also by the "potential" related to SW $\phi_0(\xi)$, and the mode energy $\delta E_k(\tau)$ not only depends on amplitude b_k (PW self-energy) but also on the phase difference between $\delta\phi_k$ and $\phi_{0,k}$; i.e., $(\theta_k - \alpha_k)$, the contribution of the latter (interaction-energy between the PW and SW) to the total energy can be either positive or negative. Therefore, apart from adjusting the relative movements of different oscillators $\delta\phi_k$, the system also needs to adjust their positions in the potential. It is remarkable that this nonlinear system is still able to self-organize itself and, from time to time, effectively add all the mode energies up to highest possible values.

Apart from those strong bursts in $\delta E(\tau)$ there are also many smaller bursts that distribute between them; correspondingly one can find smaller peaks in C_α^4 (see Fig. 4). Figure 5 gives the distribution $P(\tau_L)$; here τ_L is the interspike interval determined by the threshold $C_\alpha^4 = 0.05$. A power law $\tau_L^{-\alpha}$ is displayed with $\alpha \approx 1.34$.

IV. Mechanism and discussion.—The effect of the saddle point is of importance in the on-off CIS, just as in the imperfect synchronization in the Lorenz models [8]. However, one may ask why in the SR before the crisis transition the system does not show such synchronization-induced strong bursts, despite the fact that the unstable SW is also a saddle point. This problem can be understood from the different behaviors of the master mode ($k = 1$) before and after the transition. Figure 6 gives a plot of $\alpha_{k=1}(\tau)$, which shows a sudden change at

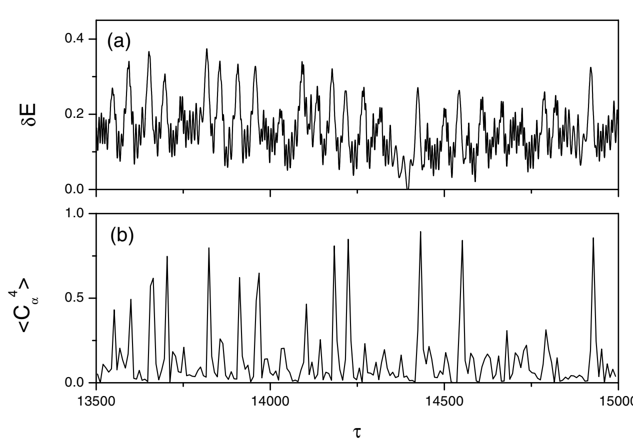

FIG. 4. (a) $\delta E(\tau)$ and (b) $\langle C_\alpha^4(\tau)\rangle$.

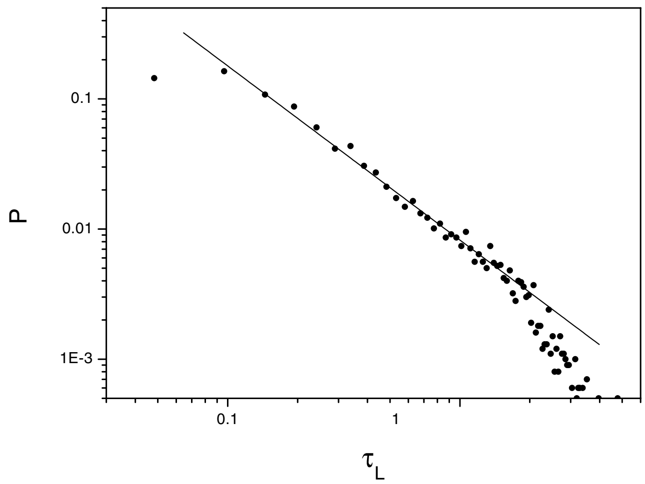

FIG. 5. Distribution function $P(\tau_L)$. The straight line gives the fit.

$\tau \approx 500$ corresponding to the transition from the transient SR to the STC. Before the transition $\alpha_{k=1}$ varies only slightly, i.e., in the (ξ, τ) frame the peak of $\delta\phi_{k=1} = b_1(\tau)\cos[\xi + \alpha_1(\tau)]$ is highly localized; hence it has little space to adjust its motion relative to other oscillators. After the transition, however, $\alpha_{k=1}$ can slip over 2π; i.e., it can be free from trapping [14]. The delocalization of the master mode allows its peak to travel a relatively long distance, providing many more possibilities to adjust the movement among the oscillators. In Fig. 6 one can see that after the crisis transition $\alpha_1(\tau)$ experiences oscillations and slips alternatively, corresponding to on and off CIS, respectively. In this case the dynamics of $\{b_k, \alpha_k\}$ is analogous to the imperfect phase synchronization in other models. For example, if in Figs. 2(b)–2(d) the mod(2π) was not taken, one can see stairslike curves where $\Delta\alpha_{1k}$ ($k = 2$–4) display plateaus, respectively, in the on stages. Normally two plateaus are connected by a fluctuating slope instead of a sudden jump.

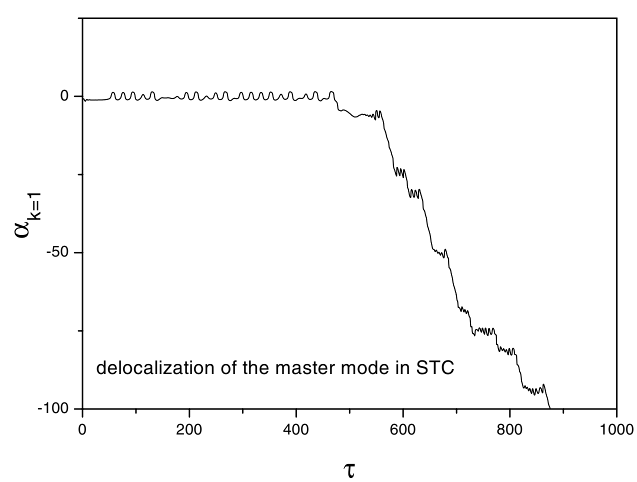

FIG. 6. State transition of $\alpha_{k=1}(\tau)$ corresponding to the transition from SR to STC.

In conclusion, we found that in a type of very turbulent STC state an on-off CIS can be established. In the on stage the system has a strong bursty manifestation; effective cooperation among oscillators in different scales brings about strong bursts in the total system energy. The interspike intervals display a power-law distribution. The present result may provide an explanation for the mechanism of energy bursts observed in different physical applications. Moreover, it is likely that the energy cascade as well as the soliton gas in the STC is also related to the on-off CIS; these problems need to be studied further.

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