ORBITAL EVOLUTION OF A SATELLITE PERTURBED BY A THIRD-BODY

Ione Vicente da Costa
Antonio Fernando Bertachini de Almeida Prado
Instituto Nacional de Pesquisas Espaciais - São José dos Campos-SP-12227-010 - Brazil
Phone (12)345-6201 - Fax (12)345-6226 - E-mail: PRADO@DEM.INPE.BR

ABSTRACT

This paper has the goal of developing a semi-analytical study of the perturbation caused in a spacecraft by a third body involved in the dynamics. One of the important applications of the present research is to calculate the effect of Lunar and Solar perturbations on high-altitude Earth satellites. There is a special interest to see under which conditions a near-circular orbit remains near-circular. The so called "critical angle of the third-body perturbation," that is a value for the inclination such that any near-circular orbit with inclination below this value remains near-circular, is discussed in detail.

INTRODUCTION

The assumptions of our model to solve the problem formulated above are very similar to the ones made in the restricted three-body problem: a) There are only three bodies involved in the system: one body with mass m_0 fixed in the origin of the reference system; a massless spacecraft in a generic orbit around this body and a third body in a circular orbit around this same central body in the plane x-y; b) The motion of the spacecraft is supposed to be a three-dimensional keplerian orbit with its orbital elements perturbed by the third body. The motion of the spacecraft is studied under the double-averaged analytical model (Broucke, 1992; Costa, 1998) with the disturbing function expanded up to the Legendre polynomial $P_8(\text{Cos}(S))$.

REVIEW OF THE LITERATURE

Several papers studied the third-body perturbation using different approaches. The majority of them studied the perturbation due to the Sun and the Moon in a satellite in orbit around the Earth. Kozai (1959) developed the main secular and long period terms of the disturbing function due to the lunisolar perturbations in terms of the orbital elements of the satellite, the Sun and the Moon. This

research would be later expanded by Musen, Bailie and Upton (1961) to include the parallactic term in the disturbing function. After that, Kozai (1962) studied the problem of secular perturbations in asteroids with high inclination and eccentricity, assuming that they are perturbed by Jupiter, which is supposed to be in a circular orbit around the Sun. In another research, Blitzer (1959) obtained estimates of the lunisolar disturbances using methods of classical mechanics, however only for the secular terms. In the sequence, Cook (1962) used the Lagrange's planetary equations to obtain expressions for the variation of the elements during a revolution of the satellite and for the rate of variation of the same elements. In that same year, Kaula (1962) derived general terms of the disturbing function for the lunisolar perturbation, using equatorial elements for the Moon, but it didn't supply a definitive algorithm for the calculations. Again Kozai (1965) approached that problem and included indirect terms of the disturbance due to the alteration of the terrestrial flattening due to those forces.

In the seventies, that subject was studied again. Giacaglia (1973) obtained the disturbing function for the disturbance of the Moon using ecliptic elements for the Moon and equatorial elements for the satellite. Secular, long and short period terms were calculated and expressed in a closed form. Kozai (1973) developed an alternative method for the calculation of the lunisolar disturbances. The disturbing function was expressed in terms of the orbital elements of the satellite and the polar geocentric coordinates of the Sun and the Moon. The secular and long period terms are derived by numerical integration and the short period terms are obtained analytically. In the following decade, Hough (1981) studied the effects of the lunisolar disturbance in orbits close to the inclinations 63,4° and 116,6° (critical inclinations from the point of view of the geopotential of the Earth) and concluded that the effects are significant in high altitudes. All those researches represent fundamental contributions in the area and they possess an analytic focus, dedicated to the derivations of equations. In the present work a more practical approach is used with the idea of complementing the existent literature. There are a few studies that appeared recently in the literature, looking for results and numeric comparisons. A good examples are the researches developed by Broucke (1992) and Prado and Costa (1998), where the general form of the disturbing function of the third body was truncated after the term of second and fourth order, respectively, in the expansion in Legendre polynomials. From this study, several conclusions are obtained for the second order model. After that, Costa (1998) expanded the order of this model to the order eight. In the present research, some of the conclusions obtained by Broucke (1992) for the second order model are verified for the fourth order model and also for the full restricted three-body problem.

THE MATHEMATICAL MODEL

This section derives the equations required by the mathematical model used during the simulations made in this research. Fig. 1 shows the situation. The main body with mass m_0 is fixed in the center of the reference system x-y. The perturbing body with mass m' is in a circular orbit with semi-major axis a' and mean motion n' (given by the expression $n'^2 a'^3 = G[m_0 + m']$) in the plane of the figure. The massless spacecraft m is in a generic three dimensional orbit which orbital elements are: a, e, i, ω, Ω and the mean motion is n (given by the expression $n^2 a^3 = Gm_0$). The figure is a view of the x-y plane, but the direction z (perpendicular to the plane shown) also exists and it is available for the motion of the spacecraft.
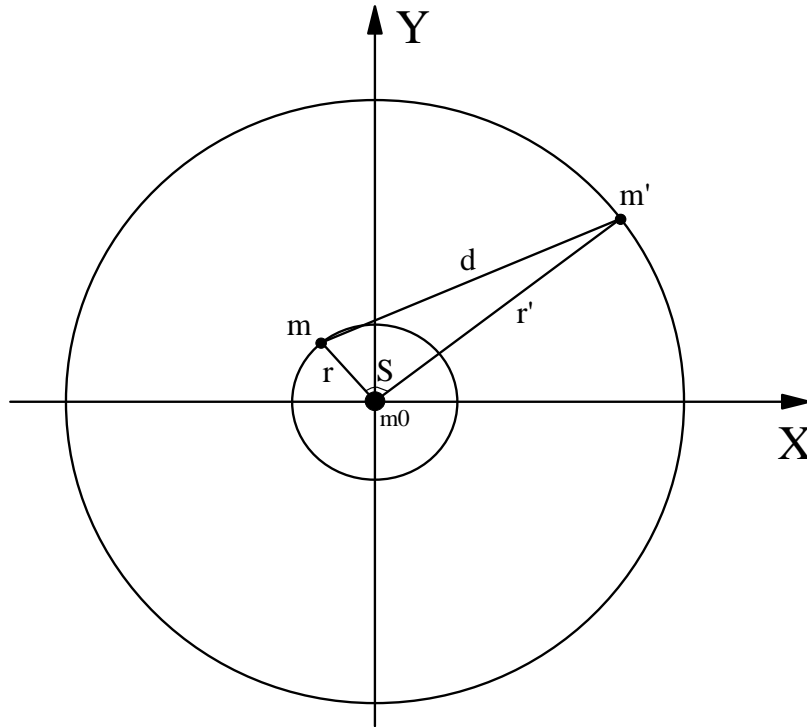
Fig. 1 - Illustration of the Third-Body Perturbation.

In this situation, the disturbing potential that the spacecraft has from the action of the disturbing body is given by (using the traditional expansion in Legendre polynomials and assuming that $r' \gg r$):

$$R = \frac{\mu'}{\sqrt{r^2 + r'^2 - 2rr'Cos(S)}} = \frac{\mu'}{r'} \sum_{n=2}^{\infty} \left(\frac{r}{r'}\right)^n P_n(Cos(S)) \tag{1}$$

For the model used in this research it is necessary to calculate the parts of the disturbing function due to P_2 to P_8 (R_2 to R_8, respectively). They are (Costa, 1998):

$$R_2 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^2 P_2(Cos(S)) = \frac{\mu' n'^2 a^2}{2}\left(\frac{a'}{r'}\right)^3\left(\frac{r}{a}\right)^2 \left(3Cos^2(S) - 1\right) \tag{2}$$

$$R_3 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^3 P_3(Cos(S)) = \frac{\mu' n'^2 a^3}{2r'}\left(\frac{a'}{r'}\right)^3\left(\frac{r}{a}\right)^3 \left(5Cos^3(S) - 3Cos(S)\right) \tag{3}$$

$$R_4 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^4 P_4(Cos(S)) = \frac{\mu' n'^2 a^4 a'^3}{8r'^5}\left(\frac{r}{a}\right)^4 \left(35Cos^4(S) - 30Cos^2(S) + 3\right) \tag{4}$$

$$R_5 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^5 P_5(Cos(S)) = \frac{\mu' n'^2 a'^3 a^5}{8r'^6}\left(\frac{r}{a}\right)^5 \left(63Cos^5(S) - 70Cos^3(S) + 15Cos(S)\right) \tag{5}$$

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$$R_6 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^6 P_6(Cos(S)) = \frac{\mu'}{r'}\frac{r^6}{r'^6}\frac{1}{16}\left(231Cos^6(S) - 315Cos^4(S) + 105Cos^2(S) - 5\right) \tag{6}$$

$$R_7 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^7 P_7(Cos(S)) = \frac{\mu'n'^2 a'^3 a^7}{16r'^8}\left(\frac{r}{a}\right)^7\left(429Cos^7(S) - 693Cos^5(S) + 315Cos^3(S) - 35Cos(S)\right) \tag{7}$$

$$R_8 = \frac{\mu'}{r'}\left(\frac{r}{r'}\right)^8 P_8(Cos(S)) =$$

$$= \frac{\mu'n'^2 a'^3 a^8}{32r'^8 r^8}\left(\frac{5315}{4}Cos^8(S) - 1883Cos^6(S) + \frac{105}{2}Cos^4(S) + 805Cos^2(S) - \frac{1085}{4}\right) \tag{8}$$

The next step is to average those quantities over the short period of the satellite as well as with respect to the distant perturbing body. The standard definition for average used in this research is:

$\langle f \rangle = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}(f)dM$, where M is the mean anomaly that is proportional to time. The results are (for

the special case considered in this research of circular orbits for the perturbing body) (Costa, 1998):

$$\langle \overline{R}_2 \rangle = \frac{\mu'a^2n'^2}{16}\left[(2 + 3e^2)(3Cos^2(i) - 1) + 15e^2 Sin^2(i)Cos(2\omega)\right] \tag{9}$$

$$\langle \overline{R}_3 \rangle = \langle \overline{R}_5 \rangle = \langle \overline{R}_7 \rangle = 0 \tag{10}$$

$$\langle \overline{R}_4 \rangle = \frac{9n'^2 a^4}{65536a'^2}\left[144 + 720e^2 + 270e^4 + (320 + 1600e^2 + 600e^4)Cos(2i) + (560 + 2800e^2 + 1050e^4)\right.$$
$$Cos(4i) + (1680e^2 + 840e^4)Cos(2\omega) + 4410e^4 Cos(4\omega) + (2240e^2 + 1120e^4)Cos(2i)Cos(2\omega) +$$
$$\left. + (3920e^2 + 1960e^4)Cos(4i)Cos(2\omega) + 5880e^4 Cos(2i)Cos(4\omega) + 1470e^4 Cos(4i)Cos(4\omega)\right] \tag{11}$$

$$\langle \overline{R}_6 \rangle = \frac{15}{256}\frac{\mu'n'^2 a'^3 a^6}{16r'^7}\{77(5 + 120e^2 + 240e^4 + 64e^6)(3Cos^2\omega Cos^4 iSen^4\omega + 3Cos^4\omega Cos^2 iSen^2\omega + Cos^6 i) +$$
$$+ 231(1 + 15e^2 - 16e^6)(5Cos^2\omega Cos^2 iSen^4\omega + 6Cos^4\omega Cos^4 iSen^2\omega - 8Cos^2\omega Cos^4 iSen^4\omega + Cos^4 iSen^6\omega +$$
$$+ Cos^6\omega Cos^2 i - 8Cos^4\omega Cos^2 iSen^2\omega + 6Cos^2\omega Cos^2 iSen^4\omega + 5Cos^4\omega Sen^2\omega) + 231(-1 + e^2)^2(1 + 8e^2) +$$
$$(5Cos^4\omega Cos^6 iSen^6\omega + Cos^6\omega Cos^4 i - 8Cos^4\omega Cos^4 iSen^2\omega + 6Cos^2\omega Cos^4 iSen^4\omega + 6Cos^4\omega Cos^2 iSen^2\omega -$$
$$- 8Cos^2\omega Cos^2 iSen^4\omega + Cos^2 iSen^4\omega + 5Cos^2\omega Sen^4\omega) + 77(1 - e^2)^3(5Cos^6\omega Cos^6 i + 15Cos^4\omega Cos^4 iSen^2\omega +$$
$$+ 15Cos^2\omega Cos^2 iSen^4\omega + 5Sen^6\omega) - 378(2 + 45e^2 + 80e^4 + 16e^6)(Cos^2\omega Cos^2 iSen^2\omega)^2 - 252(2 + 21e^2 -$$
$$- 15e^4 - 8e^6)(3Cos^2\omega Sen^2\omega + Cos^4\omega Cos^2 i - 4Cos^2\omega Cos^2 iSen^2\omega + Cos^2 iSen^4\omega + 3Cos^2\omega Cos^4 iSen^2\omega) -$$
$$- 378(-1 + e^2)^2(2 + e^2)(Cos^2\omega Cos^2 i + Sen^2\omega)^2 + 56(8 + 156e^2 + 225e^4 + 40e^6)(Cos^2\omega + Cos^2 iSen^2\omega) +$$

$$+56(8+12e^2-15e^4-5e^6)(Cos^2\omega Cos^2 i++Sen^2\omega)-\frac{16}{3}(16+168e^2+210e^4+35e^6)\} \tag{12}$$

$$\langle R_8 \rangle = \frac{\mu' n'^2 \, a'^3 a^8}{32 r'^8} \{ \frac{5315}{4}(\frac{35}{128}Cos^8(i)Sen^8(\omega)+\frac{140}{128}Cos^2(\omega)Cos^6(i)Sen^6(\omega)++210Cos^4(\omega)Cos^4(i)$$

$$Sen^4(\omega)+\frac{140}{128}Cos^6(\omega)Cos^2(i)Sen^2(\omega))\frac{5}{128}(7+280e^2+1120e^4+896e^6+128e^8)+74410(\frac{35}{128}Cos^2(\omega)$$

$$Cos^8(i)Sen^6(\omega)+\frac{75}{128}Cos^4(\omega)Cos^6(i)Sen^4(\omega)-\frac{70}{128}Cos^2(\omega)Cos^6(i)Sen^6(\omega)+\frac{5}{128}Cos^6(i)Sen^8(\omega)+$$

$$+45Cos^6(\omega)Cos^4(i)Sen^2(\omega)-120Cos^4(\omega)Cos^4(i)Sen^4(\omega)+\frac{75}{128}Cos^2(\omega)Cos^4(i)Sen^6(\omega)+\frac{5}{128}$$

$$Cos^8(\omega)Cos^2(i)-\frac{70}{128}\cdot Cos^6(\omega)Cos^2(i)Sen^2(\omega)+\frac{75}{128}Cos^4(\omega)Cos^2(i)Sen^4(\omega)+\frac{35}{128}Cos^6(\omega)$$

$$Sen^2(\omega))\frac{5}{128}(1+29e^2+50e^4-48e^6-32e^8)+\frac{186025}{2}(\frac{35}{128}Cos^4(\omega)Cos^8(i)Sen^4(\omega)+$$

$$+\frac{30}{128}Cos^6(\omega)Cos^8(i)Sen^2(\omega)-\frac{80}{128}Cos^4(\omega)Cos^6(i)Sen^4(\omega)+\frac{30}{128}Cos^2(\omega)Cos^6(i)Sen^6(\omega)-$$

$$-\frac{40}{128}Cos^6(\omega)Cos^4(i)Sen^2(\omega)+\frac{108}{128}Cos^4(\omega)Cos^4(i)Sen^4(\omega)-Sen(i)Cos^2(\omega)Cos^4(i)Sen^6(\omega)+$$

$$+\frac{3}{128}Cos^4(i)Sen^8(\omega)+\frac{30}{128}Cos^6(\omega)Cos^2(i)Sen^2(\omega)-\frac{80}{128}Cos^4(\omega)Cos^2(i)Sen^4(\omega)+$$

$$+\frac{30}{128}Cos^2(\omega)Cos^2(i)Sen^6(\omega)+\frac{35}{128}Cos^4(\omega)Sen^4(\omega)\frac{1}{128}(-1+e^2)^2(3+60e^2+80e^4)+$$

$$+37205(\frac{35}{128}Cos^6(\omega)Cos^8(i)Sen^2(\omega)+\frac{5}{128}Cos^8(\omega)Cos^6(i)-\frac{60}{128}Cos^6(\omega)Cos^6(i)Sen^2(\omega)+$$

$$+\frac{75}{128}Cos^4(\omega)Cos^6(i)Sen^4(\omega)+\frac{45}{128}Cos^6(\omega)Cos^4(i)Sen^4(\omega)-\frac{120}{128}Cos^4(\omega)Cos^4(i)Sen^4(\omega)+$$

$$+\frac{45}{128}Cos^2(\omega)Cos^4(i)Sen^6(\omega)+\frac{75}{128}Cos^4(\omega)Cos^2(i)Sen^4(\omega)-60Cos^2(\omega)Cos^2(i)Sen^6(\omega)+$$

$$+\frac{5}{128}Cos^4(i)Sen^8(\omega)+\frac{35}{128}Cos^2(\omega)Sen^6(\omega)\frac{5}{128}(1-e^2)^3(1+10e^2)+\frac{5315}{4}(\frac{35}{128}Cos^8(\omega)Cos^8(i)+$$

$$+\frac{140}{128}Cos^6(\omega)Cos^6(i)Sen^2(\omega)+\frac{168}{128}Cos^5(\omega)Cos^5(i)Sen^3(\omega)+$$

$$+\frac{210}{128}Cos^4(\omega)Cos^4(i)Sen^4(\omega)+\frac{140}{128}Cos^2(\omega)Cos^2(i)Sen^6(\omega)+\frac{35}{128}Sen^8(\omega))\frac{35}{128}(-1+e^2)^4$$

$$-1883(\frac{5}{16}Cos^6(i)Sen^6(\omega)+\frac{15}{16}Cos^2(\omega)Cos^4(i)Sen^4(\omega)+\frac{15}{16}Cos^4(\omega)$$

$$Cos^2(i)Sen^2(\omega)+\frac{5}{16}Cos^6(\omega))\frac{5}{128}(8+309e^2+1170e^4+848e^6+96e^8)-28245(\frac{5}{16}Cos^2(\omega)$$

$$Cos^6(i)Sen^4(\omega) + \frac{6}{16}Cos^4(\omega)Cos^4(i)Sen^2(\omega) - \frac{8}{16}Cos^2(\omega)Cos^4(i)Sen^4(\omega) + \frac{1}{16}Cos^4(i)Sen^6(\omega) +$$

$$+\frac{1}{16}Cos^6(\omega)Cos^2(i) - \frac{8}{16}Cos^4(\omega)Cos^2(i)Sen^2(\omega) + \frac{6}{16}Cos^2(\omega)Cos^2(i)Sen^4(\omega) +$$

$$+\frac{5}{16}Cos^4(\omega)Sen^2(\omega))\frac{1}{128}(8+199e^2+213e^4-340e^6-80e^8-28245(\frac{5}{16}Cos^4(\omega)Cos^6(i)Sen^6(\omega) +$$

$$+\frac{1}{16}Cos^6(\omega)Cos^4(i) - \frac{8}{16}Cos^4(\omega)Cos^4(i)Sen^2(\omega) + \frac{6}{16}Cos^2(\omega)Cos^4(i)Sen^4(\omega) +$$

$$+\frac{6}{16}Cos^4(\omega)Cos^2(i)Sen^2(\omega) - \frac{8}{16}Cos^2(\omega)Cos^2(i)Sen^4(\omega) + \frac{1}{16}Cos^2(i)Sen^4(\omega) + \frac{5}{16}Cos^2(\omega)$$

$$Sen^4(\omega))\frac{1}{128}(-1+e^2)^2(8+105e^2+30e^4) - 1883(\frac{5}{16}Cos^6(\omega)Cos^6(i) + \frac{15}{16}Cos^4(\omega)Cos^4(i)Sen^2(\omega) +$$

$$+\frac{15}{16}Cos^2(\omega)Cos^2(i)Sen^4(\omega) + \frac{5}{16}Sen^6(\omega))\frac{5}{128}(-1+e^2)^3(8+3e^2) + \frac{105}{2}[\frac{3}{8}(Cos^2(\omega) +$$

$$+Cos^2(i)Sen^2(\omega))^2](48+1744e^2+6063e^4+3900e^6+400e^8)\frac{1}{128}+315(\frac{3}{8}Cos^2(\omega)Sen^2(\omega) +$$

$$+\frac{1}{8}Cos^4(\omega)Cos^2(i) - \frac{4}{8}Cos^2(\omega)Cos^2(i)Sen^2(\omega) + \frac{1}{8}Cos^2(i)Sen^4(\omega) + \frac{3}{8}Cos^2(\omega)Cos^4(i)Sen^2(\omega)) +$$

$$+\frac{1}{128}(16+288e^2+41e^4-295e^6-50e^8) + \frac{315}{2048}(Sen^2(\omega)+Cos^2(\omega)Cos^2(i))^2(-1+e^2)^2(48+80e^2 +$$

$$+15e^4) + \frac{805}{256}(64+2032e^2+6104e^4+3605e^6+350e^8)(Cos^2(\omega)+Cos^2(i)Sen^2(\omega)) +$$

$$+\frac{805}{256}(Sen^2(\omega)+Cos^2(i)Cos^2(\omega))(64+272e^2-56e^4-245e^6-35e^8) - \frac{1085}{4}\} \qquad (13)$$

After that, the next step is to obtain the equations of motion of the spacecraft. They come from the Lagrange's planetary equations in the form that depends on the derivatives of the disturbing function R with respect to the keplerian elements. Those equations are (Taff, 1985):

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial R}{\partial M_0} \qquad (14)$$

$$\frac{de}{dt} = \frac{1-e^2}{na^2e}\frac{\partial R}{\partial M_0} - \frac{(1-e^2)^{1/2}}{na^2e}\frac{\partial R}{\partial \omega} \qquad (15)$$

$$\frac{d\omega}{dt} = \frac{-Cos(i)}{Sin(i)[\mu a(1-e^2)]^{1/2}}\frac{\partial R}{\partial i} + \frac{(1-e^2)^{1/2}}{na^2e}\frac{\partial R}{\partial e} \qquad (16)$$

$$\frac{di}{dt} = \frac{Cos(i)}{Sin(i)[\mu a(1-e^2)]^{1/2}}\frac{\partial R}{\partial \omega} - \frac{1}{Sin(i)[\mu a(1-e^2)]^{1/2}}\frac{\partial R}{\partial \Omega} \qquad (17)$$

$$\frac{d\Omega}{dt} = \frac{1}{Sin(i)[\mu a(1-e^2)]^{1/2}}\frac{\partial R}{\partial i} \qquad (18)$$

$$\frac{dM_0}{dt} = -\frac{(1-e^2)}{na^2e}\frac{\partial R}{\partial e} - \frac{2}{na}\frac{\partial R}{\partial a} \tag{19}$$

Only second order equations (similar to the ones obtained by Broucke, 1992) are shown here, due to space limitations and the fact that some conclusions are obtained direct from the equations, for the second order model.

$$\frac{da}{dt} = 0 \tag{20}$$

$$\frac{de}{dt} = \frac{15\mu'n'^2\,e\sqrt{1-e^2}}{8n}Sen^2(i)Sen(2\omega) \tag{21}$$

$$\frac{di}{dt} = \frac{-15\mu'n'^2\,e^2}{16n\sqrt{1-e^2}}Sen(2i)Sen(2\omega) \tag{22}$$

$$\frac{d\omega}{dt} = \frac{3\mu'n'^2}{8n\sqrt{1-e^2}}\Big[(5Cos^2i - 1 + e^2) + 5(1 - e^2 - Cos^2i)Cos(2\omega)\Big] \tag{23}$$

$$\frac{d\Omega}{dt} = \frac{3\mu'n'^2\,Cosi}{8n\sqrt{1-e^2}}\Big[5e^2Cos(2\omega) - 3e^2 - 2\Big] \tag{24}$$

$$\frac{dM_0}{dt} = \frac{-\mu'n'^2}{8n}\Big[(3e^2 + 7)(3Cos^2i - 1) + 15(1 + e^2)Sen^2iCos^2\omega\Big] \tag{25}$$

An important property for the averaged methods is that the semi-major axis always remains constant. This fact occurs because, after the averaging, the disturbing function does not depend on M_0 and equation 14 shows that, under this circumstance, $\frac{da}{dt} = 0$.

RESULTS

In this section some results are shown related to the third body perturbation problem. This section is divided into several sub-sections to show clearly several aspects of the problem.

The Near-Circular Orbits and the Critical Inclination

One of the most important properties of the third body perturbation is the existence of a critical value for the inclination between the perturbed and the perturbing body. This critical inclination is related to the stability of near-circular orbits. The problem considered is to discover under what conditions a spacecraft that starts in a near-circular orbit around the main body remains in a near-circular orbit after some time. The answer for this question depends on the initial inclination i_0. There is a specific critical value such that if the inclination is higher than that the eccentricity increases and the near-

circular orbit becomes very elliptic. Alternatively, if the inclination is lower than this critical value the orbit stays nearly circular.

The problem considering an exact circular orbit ($e = 0.0$) has to be studied separately. The problem of near-circular orbits is considered very important because usually a spacecraft that is nominally in a circular orbit has perturbations from other sources that makes its eccentricity to move away from the nominal value 0.0.

In the double-averaged second-order model this critical situation occurs when $Cos^2(i) = 0.60$ ($i = 39.2315$ degree). The behavior of the inclination and the eccentricity with time is studied for near-circular orbits covering a large range of initial inclination ($0° < i_0 \leq 90°$) for several degrees of truncation of the Legendre Polynomial. Figs. 2 to 4 show the results. For those simulations the initial orbit used always have keplerian elements $a = 0.1$, $e_0 = 0.01$, $\omega = \Omega = 0$. The initial inclination i_0 vary as shown in the figures. Remember that the time is defined such that the period of the perturbing body is 2π. In that way 1000 units of time in those figures correspond to about 160 orbits of the perturbing body.

The results show that for values of the initial inclination i_0 below the critical angle ($i_0 < 35°$) the eccentricity oscillates with a very small amplitude (less then 0.02 in most of the cases) that decreases fast when i_0 decreases. The inclination remains constant in this situation. For values of i_0 around the critical value ($38° < i_0 < 43°$) it is possible to see that the eccentricity oscillates with a larger amplitude (about 0.3) that increases with the increase of i_0. The inclination has a very characteristic behavior in this region of i_0. For values of i_0 slightly below the critical angle the inclination stays close to i_0 with an oscillation of small amplitude and large period. For values of i_0 slightly above the critical value the inclination starts at i_0, decreases until the critical value and then it returns to its original value i_0. For values of i_0 well above the critical value ($i_0 \geq 47°$) the eccentricity oscillates with increasing amplitudes until it reaches an amplitude of 1.0 in the case $i_0 = 90°$. The inclination keeps its characteristic behavior of starting at i_0, decreasing to the critical value and then returning to its original value i_0. Figs. 2 to 4 also shows that this behavior repeats itself in an endless cycle. The time required to reach the critical value decreases when i_0 increases. This phenomenon does not occur only for the case $i_0 = 90°$, when the inclination remains constant and the eccentricity goes to 1.0. Those results show that the critical angle is not a sharp separation between stable and unstable near-circular orbits. This region has a gradual transition where the eccentricity oscillates with an amplitude that increases fast with i_0, reaching the value 1.0 only in the case $i_0 = 90°$. When more terms of the expansion are used, some perturbations of shorter periods appear in the plots, but the general characteristics of the results stay the same. The practical application of those results is that only near-circular orbits with inclination lower than the critical value are useful, since above this value the orbit looses its characteristics of near-circularity.

The Circular Orbits

Directly from the equations of motion for the double-averaged second order model (equations 20 to 25) it is possible to identify the existence of circular solutions for this problem. It means that, in the ideal case of an orbit that starts with eccentricity zero, its eccentricity remains always zero. This occurs because the right-hand side of the equation for the time derivative of the eccentricity is zero

(it is proportional to the eccentricity). Another property of those orbits is that the inclination is also constant for the same reason (the time derivative of the inclination is proportional to the square of the eccentricity).

This is not true for the higher-order models. The expression for the derivatives has terms independent of the eccentricity that generate a term with the eccentricity in the denominator for the expression for $\dfrac{de}{dt}$. This fact does not allow the right-hand side of this equation to vanish. The same occurs for the inclination, since its variation also depends on the same derivatives.

The evolution of these two quantities (eccentricity and inclination) under the full unaveraged three-body problem was studied. It shows that the circular solutions with constant inclination do not exist in this more realistic model. The eccentricity oscillates with large amplitude (about 0.7). The inclination remains constant most of the time, but from time to time it decreases to the value of the critical inclination and then it returns to its initial value. The minimums in inclination occur in the same time with the maximums in eccentricity.

The general conclusion is that the circular solutions with constant inclination appear due to the truncation of the Legendre polynomial in terms of second-order and it is not a physical phenomenon.

The Equatorial Orbits

Another property of this system that comes directly from the inspection of the equations of motion is the existence of equatorial orbits. It means that if an orbit starts with $i_0 = 0$, the inclination and eccentricity remain constant and the orbits remain in the equatorial plane. In the second-order model this property is evident from the equations of motion. If $i_0 = 0$, then the right-hand sides of the expressions for $\dfrac{de}{dt}$ and $\dfrac{di}{dt}$ are also zero, because they are proportional to $\mathrm{Sin}^2(i)$ and $\mathrm{Sin}(2i)$, respectively.

The higher-order models has terms in $\mathrm{Sin}(i)$ in the denominator of the expressions for $\dfrac{de}{dt}$ and $\dfrac{di}{dt}$. This is due to the existence of several terms independent of the inclination in the expression for the derivatives used to obtain the equations of motion. The numerical integration of the full unaveraged model shows the existence of equatorial solutions also in this more general model. The inclination remains zero all the time and the eccentricity has only a short period oscillation with very small amplitude (about 0.05 for the case $e_0 = 0.5$).

CONCLUSIONS

This paper develops a semi-analytical mathematical model to study the third-body perturbation, using the double-averaged technique and expanding the perturbation up to eight order. The results show in detail the behavior of the orbits with respect to the initial inclination and the rule of the critical inclination in the stability of near-circular orbits. They show that this critical value is a transition region where the eccentricity has an oscillation that increases in amplitude. It is also

visible that when more terms of the expansion are used, some perturbations of shorter periods appear in the plots, but the general characteristics of the results stay the same. It is also shown the existence of equatorial solutions and the non-existence of circular solutions in the unaveraged problem.

ACKNOWLEDGEMENTS

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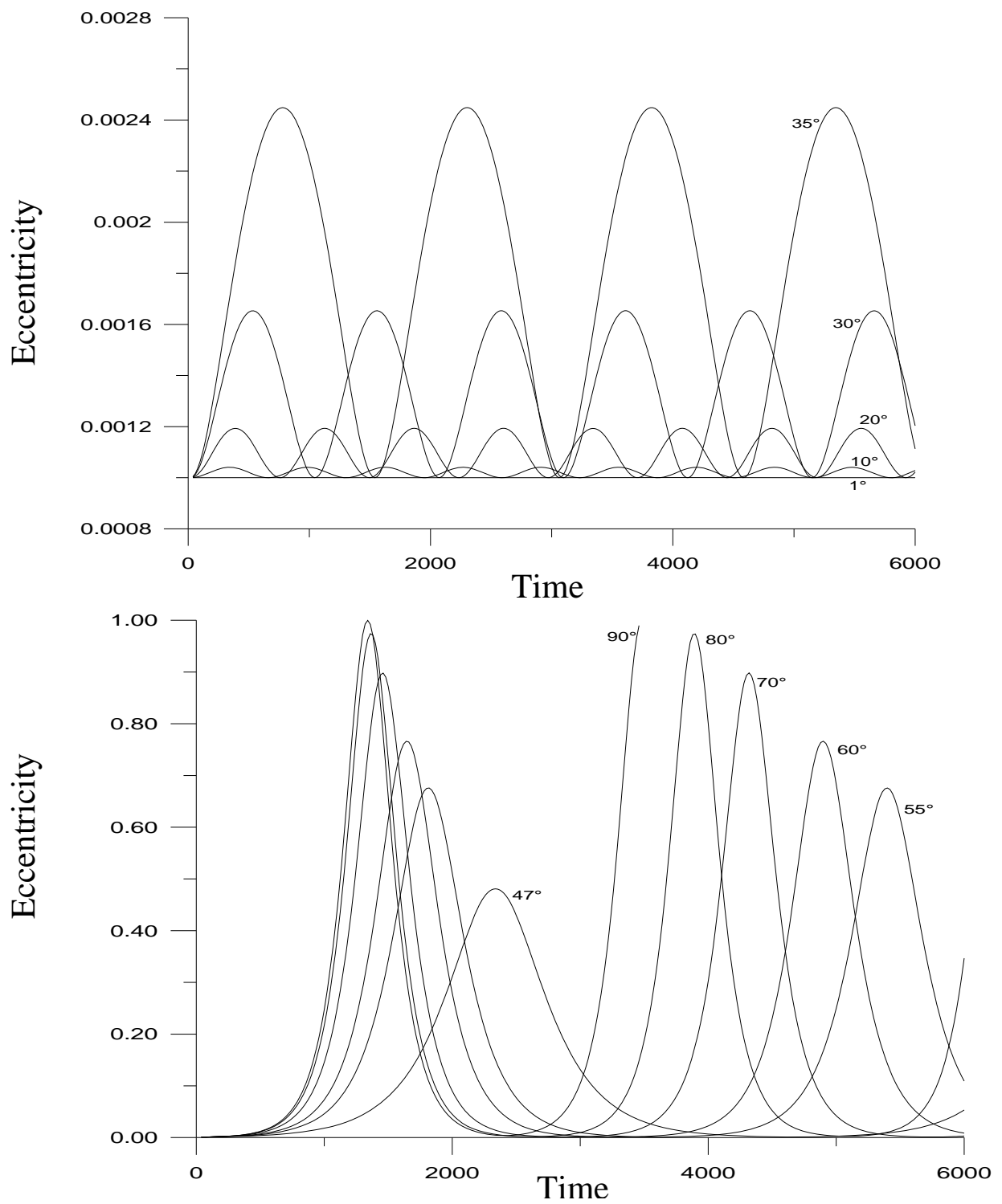
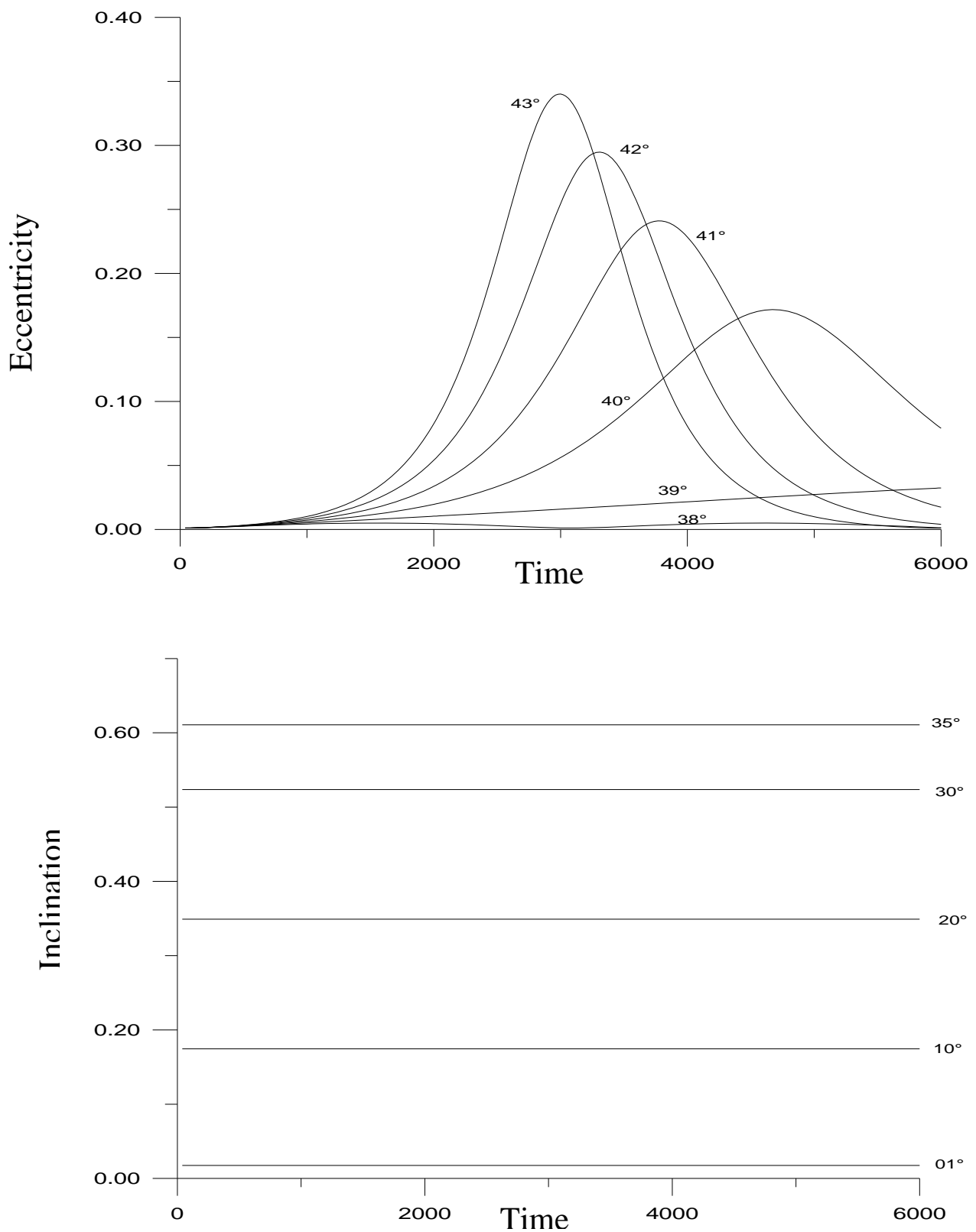Fig 2 – Eccentricity and Inclination (Legendre Polynomial truncated in the fourth order).

Fig 2 (cont) – Eccentricity and Inclination (Legendre Polynomial truncated in the fourth order).
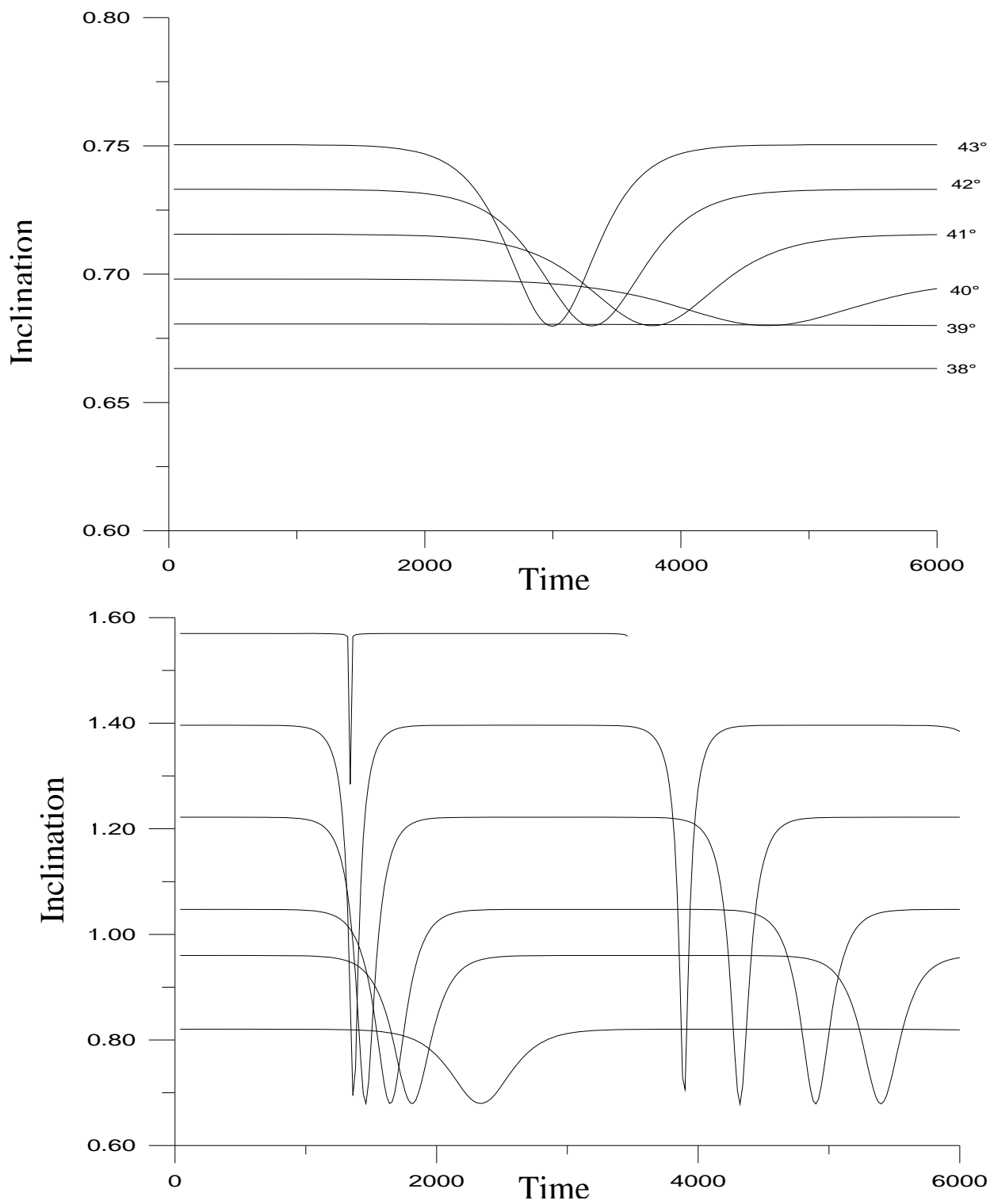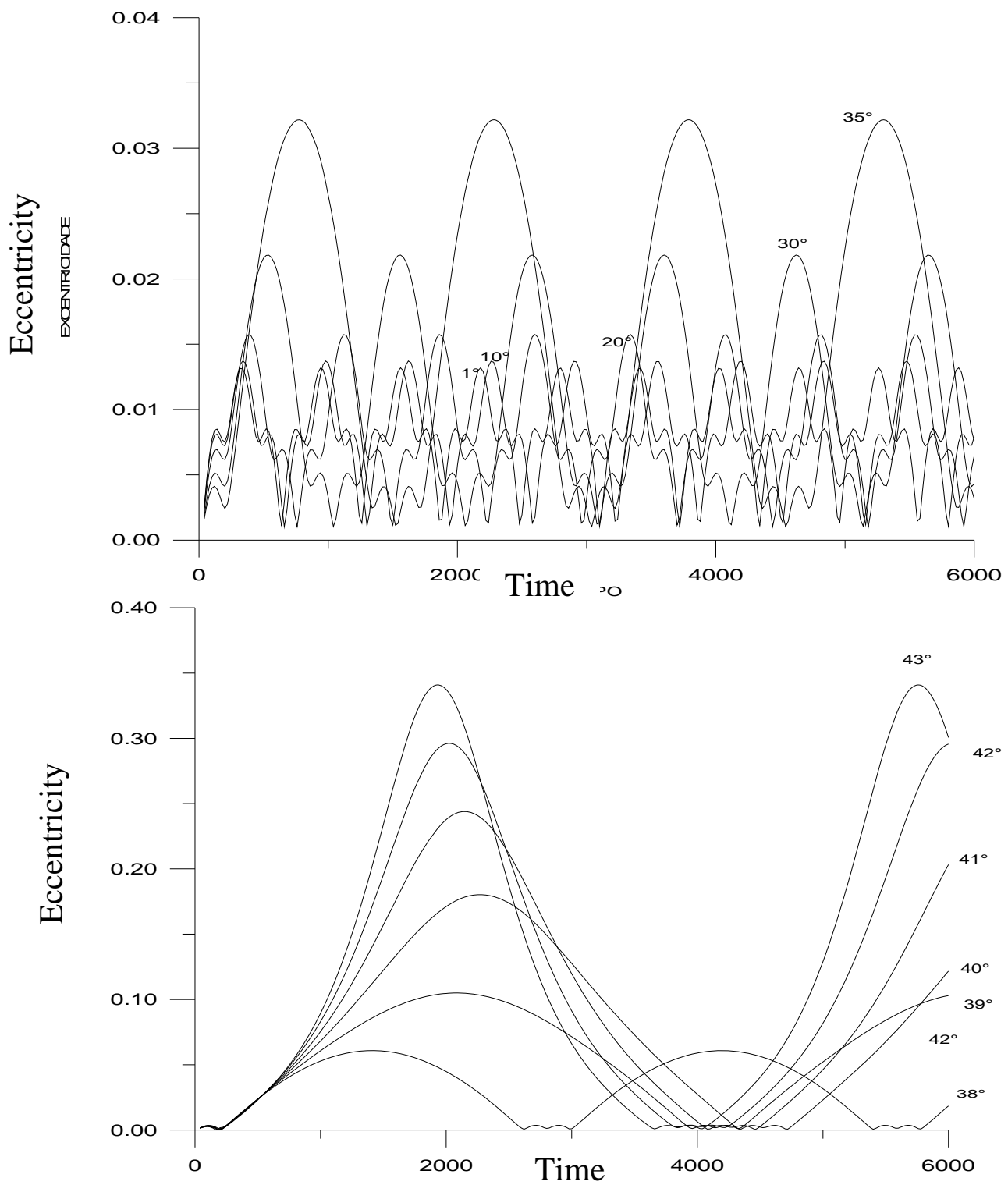
Fig 2 (cont) – Eccentricity and Inclination (Legendre Polynomial truncated in the fourth order).

Fig 3 – Eccentricity and Inclination (Legendre Polynomial truncated in the sixth order).
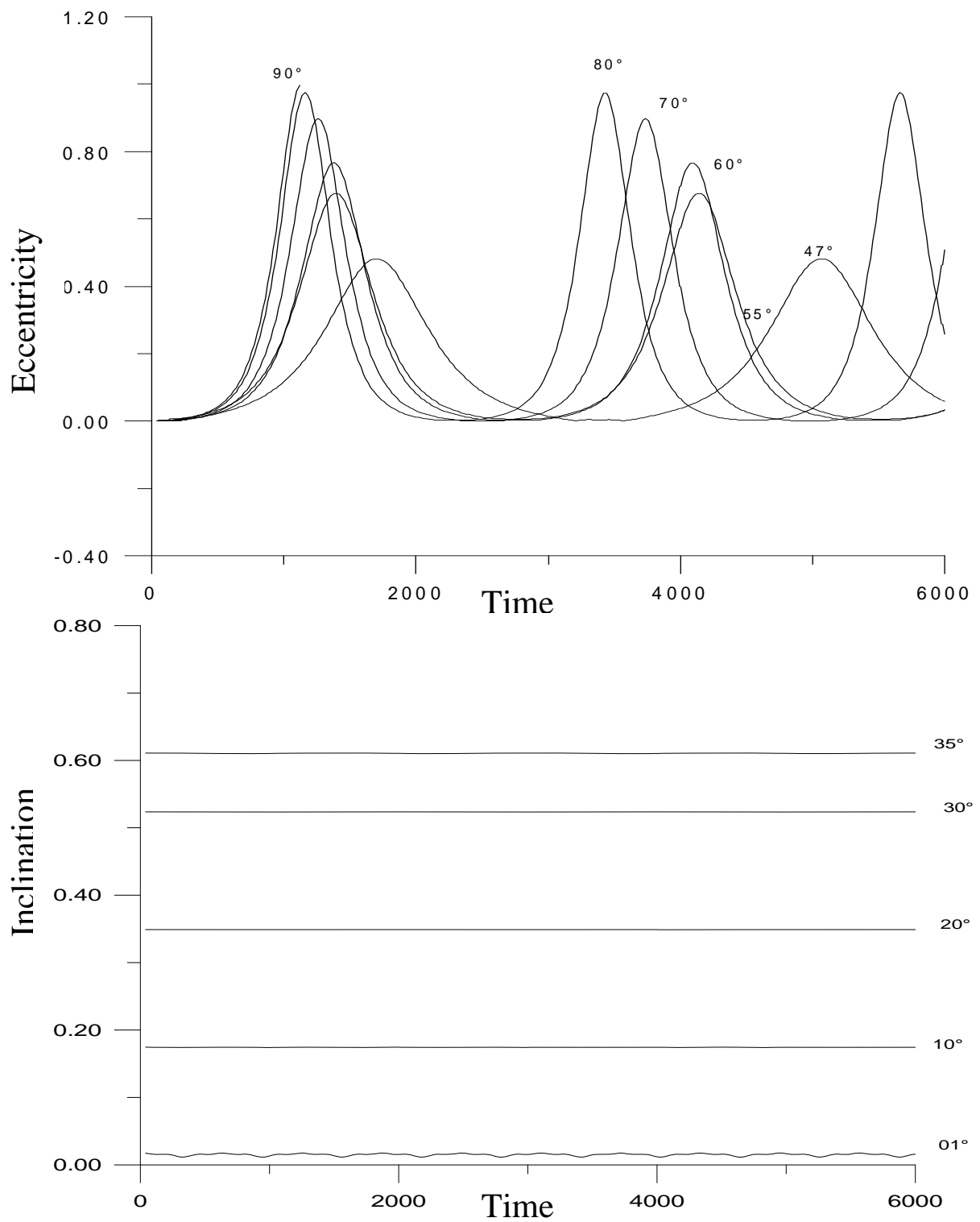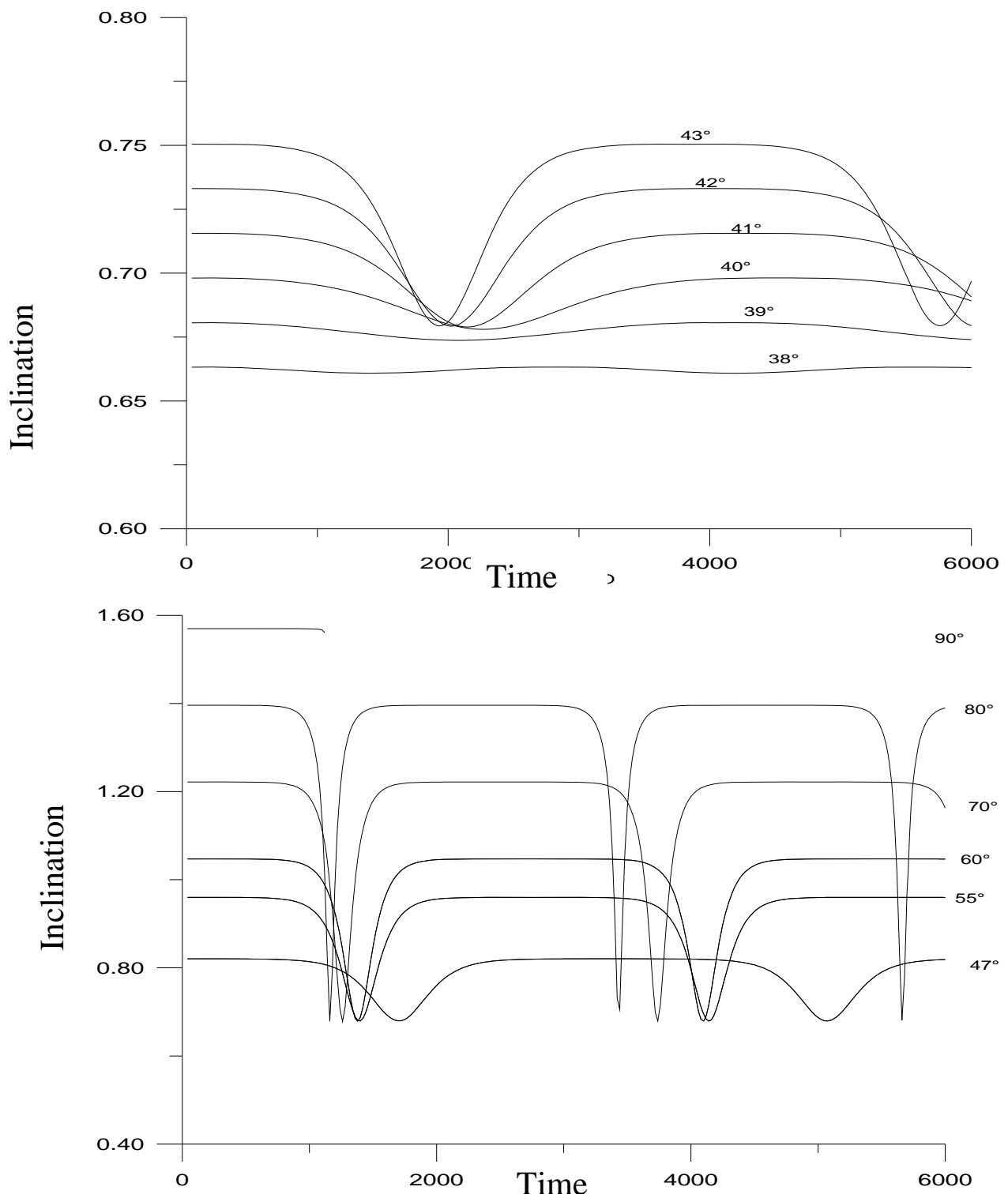
Fig 3 (cont) – Eccentricity and Inclination (Legendre Polynomial truncated in the sixth order).
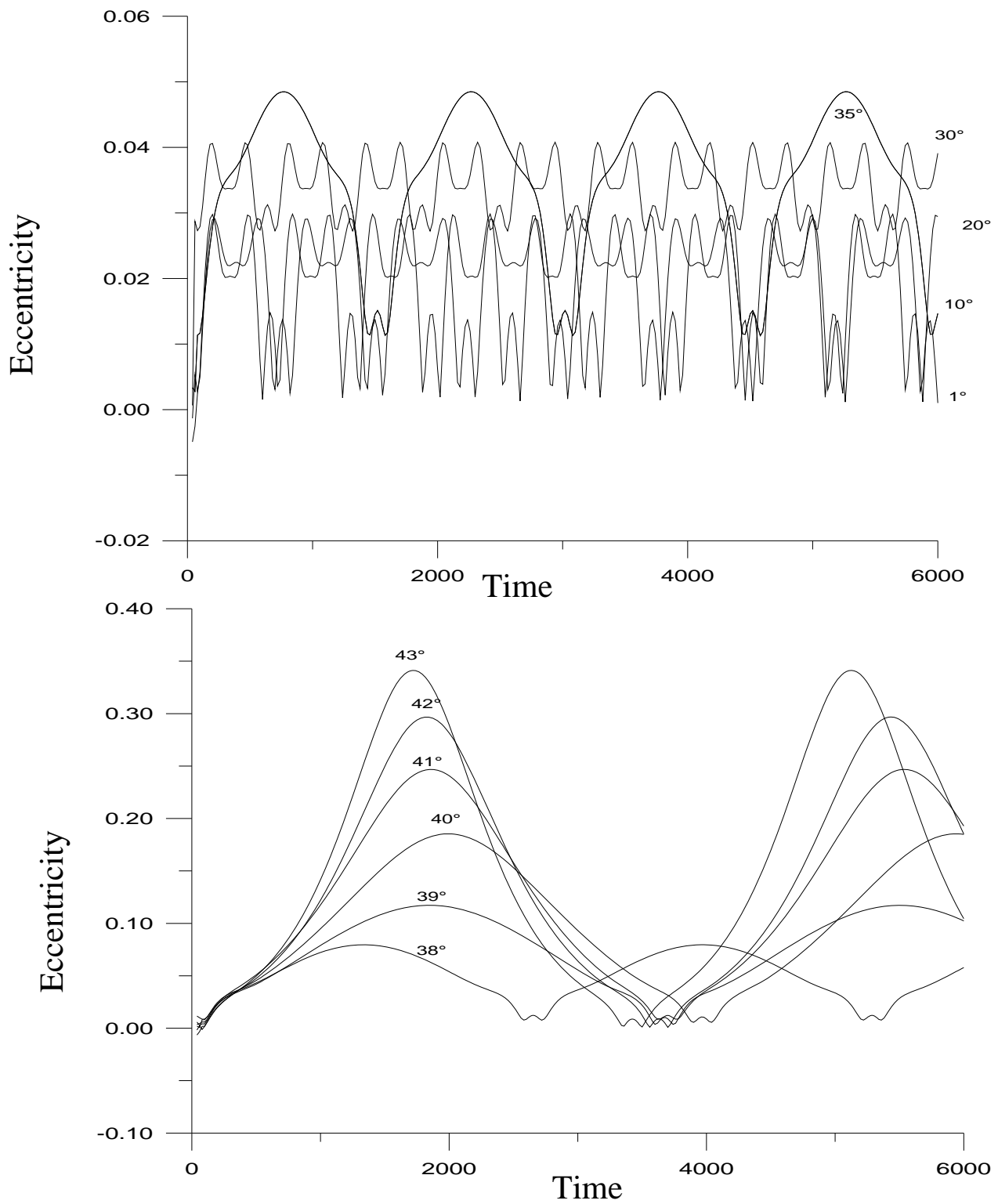
Fig 3 (cont) – Eccentricity and Inclination (Legendre Polynomial truncated in the sixth order).

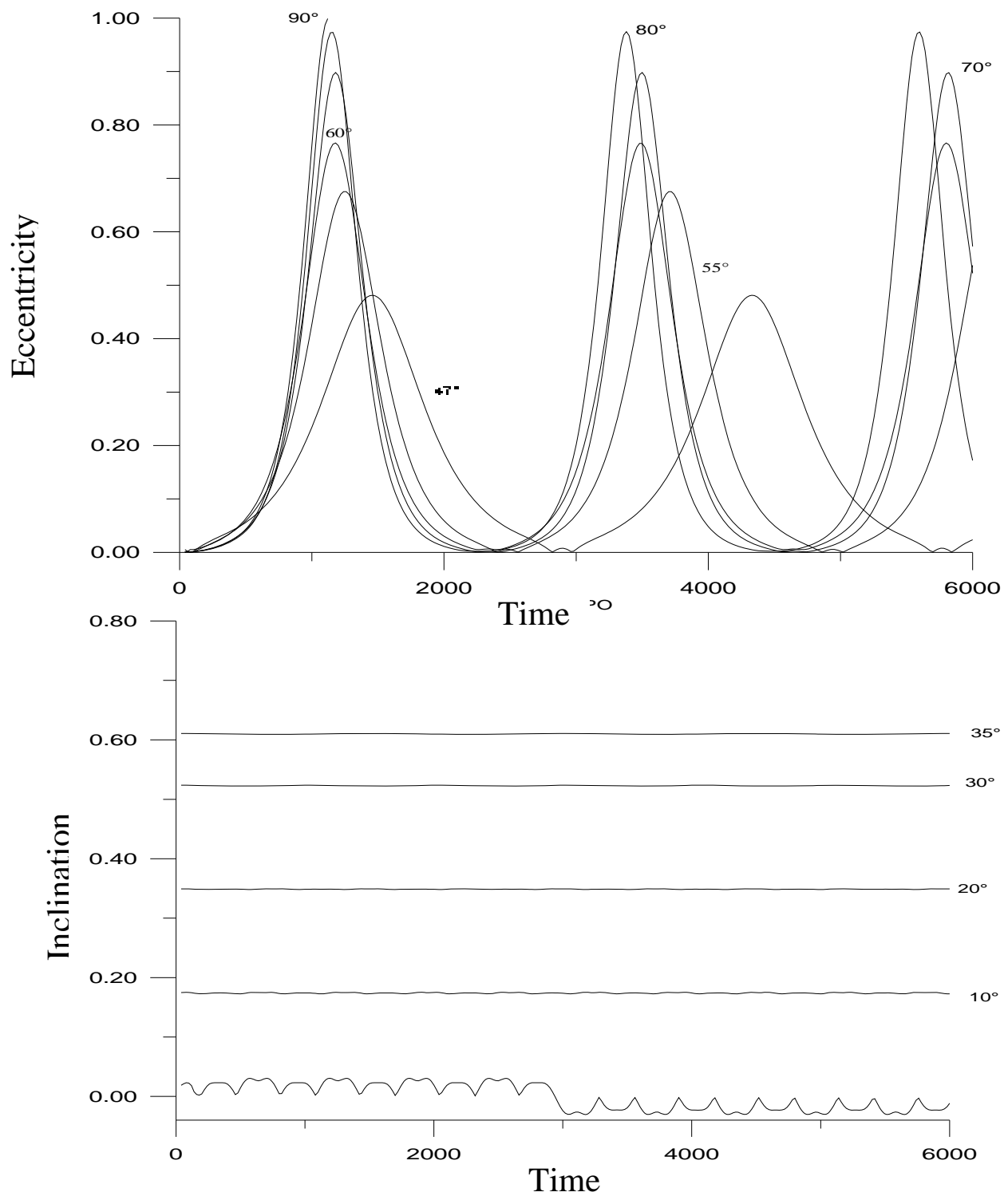Fig 4 – Eccentricity and Inclination (Legendre Polynomial truncated in the eighth order).
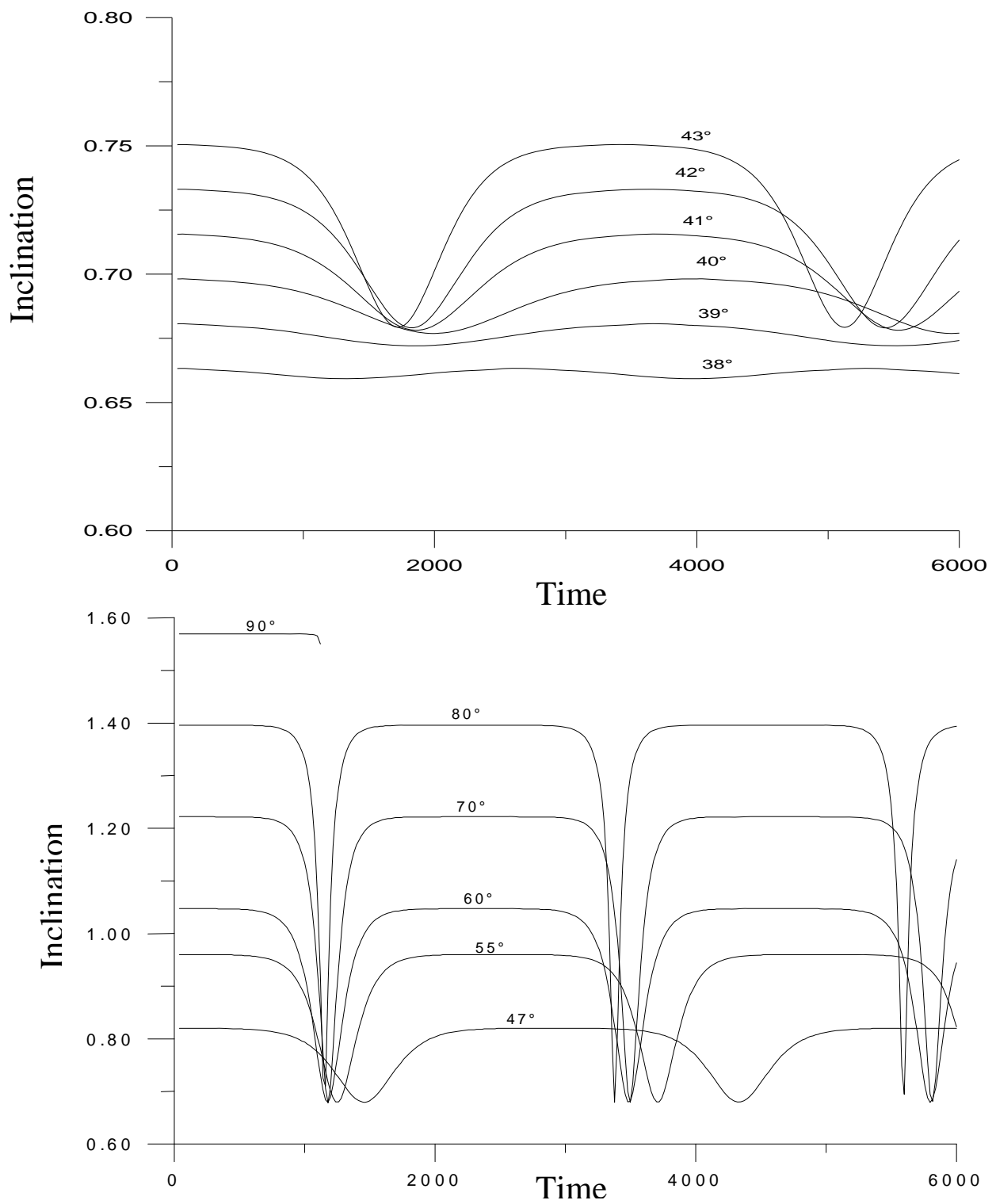
Fig 4 (cont) – Eccentricity and Inclination (Legendre Polynomial truncated in the eighth order).

Fig 4 (cont) – Eccentricity and Inclination (Legendre Polynomial truncated in the eighth order).