Dynamics of electron cyclotron current drive

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Abstract. Simple dynamical models are needed for examining the efficiency and time dependence of electron cyclotron current drive in different tokamak operating conditions. With this application in view, this work presents a fully relativistic fluid model for the energetic beam of current-carrying electrons. The model is readily applied to the study of instabilities in high-power electron cyclotron current drive experiments, indicating an additional loss mechanism that may reduce the drive efficiency.

1. Introduction

Electron cyclotron current drive (ECCD) is a powerful and well established technique with demonstrations of both full-current sustainment and localized current drive, as well as verification of the theoretical physics basis [1, 2, 3]. Usually, modeling of ECCD is carried out using complex quasi-linear three-dimensional Fokker-Planck codes [4, 5] although they are not convenient for examining the time evolution of different operational scenarios. For this application simpler dynamical models are needed, based either on a method of moments or on the Green's-function formulation (Langevin equations, adjoint method) [5]. The Green's method is limited to the linear regime, i.e. close to Maxwellian plasma, breaking down for high power densities when an intense electron beam is created [6]. Alternatively, the method of moments or fluid model depends on the adoption of a proper distribution function for the energetic electrons, thus avoiding the solution of a two-dimensional momentum space problem. The fluid model leads to results valid in the strong radio frequency (RF) regime.

In this paper a fluid model for the beam of current-carrying electrons is presented, similar to a previous work [7] but representing the beam electrons by a fully relativistic bi-Maxwellian distribution with a possible loss-cone (instead of a mono-energetic distribution) and using a definite form for the RF diffusion coefficient. This previous work, as well as much of the initial work [8] on RF current drive, considered hypothetical waves leading to idealized results. To keep the model simple, trapped particle and radial transport effects are neglected. These phenomena can considerably affect the efficiency of ECCD [2, 3] but their elimination allows studying the dynamical aspects of the problem by means of a zero-dimensional model (one expects to include these effects in a future one-dimensional formulation). In the present model both the effects of a time varying inductive electric field and quasi-linear diffusion by RF waves are included. The model is firstly developed for a general diffusion coefficient and then specialized for the case of electron cyclotron waves. In particular, the small gyro-radius limit of the diffusion coefficient is used.

Using the fully relativistic anisotropic distribution function of the beam electrons, and the quasi-linear approximation of the diffusion coefficient for a narrow spectrum of electron cyclotron waves, the evaluation of the power and force components per unit volume acting on the energetic electrons is reduced to single integrals. Then, the equations of motion can be solved looking for different equilibrium solutions or forms of time-dependent operation. In the steady-state one obtains the figure of merit for ECCD. For perpendicular injection the current driven by electron cyclotron waves vanishes and all the absorbed RF power goes to heating the beam electrons in the perpendicular direction. Initially, the figure of merit increases with the parallel refractive index $n_{\parallel}$, as indicated by experiments [3], showing a continuous transition from perpendicular to near parallel injection. An unexpected result occurs for a critical value of $n_{\parallel}$, assuming constant efficiency, when the beam equilibrium changes from a simple bi-Maxwellian to a loss-cone distribution. These strongly anisotropic equilibrium solutions are readily used to examine the characteristics of instabilities that may introduce loss of energetic particles in high-power ECCD experiments [2, 6]. It is shown that the whistler instability is excited in the background plasma by the RF driven anisotropic electron beam, possibly limiting the efficiency of current drive.

In the following sections the formulation of the method discussed above is outlined and the main results are briefly presented. In the near future the fluid model will be implemented in a set of circuit equations which allow dynamic changes in the refracting index and power injected, exploring the interaction of the RF driven current with the inductive electric field.

2. Moments of the Fokker-Planck equation

Neglecting toroidal, trapped particle and radial transport effects the Fokker-Planck equation for energetic electrons is

$$\frac{\partial f}{\partial t} = -\nabla_p \cdot \left(\overrightarrow{\Gamma}_c + \overrightarrow{\Gamma}_E + \overrightarrow{\Gamma}_{RF}\right) = -\nabla_p \cdot \overrightarrow{\Gamma}, \tag{1}$$

where $\overrightarrow{\Gamma}_c$ is the collisional flux, and the driving fluxes due to both a slow varying inductive electric field and diffusion by RF waves are given, respectively, by

$$\begin{aligned} \overrightarrow{\Gamma}_E &= -eE_\parallel f \widehat{e}_\parallel, \\ \overrightarrow{\Gamma}_{RF} &= -\overline{\overline{D}}_{RF} \cdot \nabla_p f. \end{aligned} \tag{2}$$

A general moment is obtained multiplying the Fokker-Planck equation by a momentum-dependent quantity $\Phi\left(\overrightarrow{p}\right)$ and integrating over all momentum space

$$\frac{\partial}{\partial t}\left(n\left\langle \Phi \right\rangle\right) = \int \overrightarrow{\Gamma} \cdot \nabla_p \Phi\left(\overrightarrow{p}\right) d^3p, \tag{3}$$

where it is assumed that f vanishes sufficiently fast as $\overrightarrow{p} \rightarrow \infty$ so that all moments of interest exist. For $\Phi = 1$ this leads to the conservation of the number density of energetic electrons, $\partial n / \left(\partial t\right) = 0$. The rates of change of the kinetic energy $m_e c^2 \left(\gamma - 1\right)$ and momentum $\overrightarrow{p} = m_e c \gamma \overrightarrow{\beta}$ give the equations of motion

$$\begin{aligned} \frac{\partial}{\partial t}\left\langle \gamma - 1 \right\rangle &= -\nu_c \left\langle \frac{\gamma}{\left(\gamma^2-1\right)^{1/2}} \right\rangle + \frac{P_d}{n m_e c^2}, \\ \frac{\partial}{\partial t}\left\langle \gamma \overrightarrow{\beta} \right\rangle &= -\nu_c \left\langle \frac{\left(1+Z+\gamma\right)}{\left(\gamma^2-1\right)^{3/2}} \overrightarrow{\beta} \right\rangle + \frac{\overrightarrow{F}_d}{n m_e c}, \end{aligned} \tag{4}$$

where P_d and $\overrightarrow{F}_d$ are the volumetric densities of driven power and force, and ν_c is the collision frequency (normalized to the speed of light) of energetic electrons colliding with a thermal background of electrons and ions of charge Z (e' and e designate the energetic and plasma electrons, respectively, and $\ln \Lambda_{e'e}$ is the Coulomb logarithm):

$$\nu_c = \frac{n_e e^4 \ln \Lambda_{e'e}}{4\pi \epsilon_0^2 m_e^2 c^3}. \tag{5}$$

The above equations of motion correspond, assuming a mono-energetic beam and neglecting the external sources, to the Langevin equations which describe the slowing down of energy and momentum of test electrons [8].

In terms of the driving fluxes the volumetric densities of driven power and force are given by

$$\begin{aligned} P_d &= c\int \left(\overrightarrow{\Gamma}_E + \overrightarrow{\Gamma}_{RF}\right) \cdot \overrightarrow{\beta} d^3p = -neE_\parallel c \left\langle \beta_\parallel \right\rangle - c\int \left(\overrightarrow{\beta} \cdot \overline{\overline{D}}_{RF} \cdot \nabla_p f\right) d^3p, \\ \overrightarrow{F}_d &= \int \left(\overrightarrow{\Gamma}_E + \overrightarrow{\Gamma}_{RF}\right) d^3p = -neE_\parallel \widehat{e}_\parallel - \int \left(\overline{\overline{D}}_{RF} \cdot \nabla_p f\right) d^3p. \end{aligned} \tag{6}$$

These quantities must satisfy the macroscopic constraint [9]

$$P_d = c\overrightarrow{F}_d \cdot \left\langle \overrightarrow{\beta} \right\rangle. \tag{7}$$

3. Quasi-linear RF diffusion dyadic

Using cylindrical coordinates in momentum space one has $\overrightarrow{p} = (p_\perp \cos\phi)\,\widehat{x} + (p_\perp \sin\phi)\,\widehat{y} + p_\parallel \widehat{z} = p_\perp \widehat{e}_\perp + p_\parallel \widehat{e}_\parallel$ and the gyro-averaged quasi-linear diffusion dyadic is

$$\overline{\overline{D}}_{RF} = D_\perp \widehat{e}_\perp \widehat{e}_\perp + D_\wedge \left(\widehat{e}_\perp \widehat{e}_\parallel + \widehat{e}_\parallel \widehat{e}_\perp\right) + D_\parallel \widehat{e}_\parallel \widehat{e}_\parallel. \tag{8}$$

Substituting in the expressions of the power and force per unit volume yields

$$\begin{aligned} P_d &= -neE_\parallel c \left\langle \beta_\parallel \right\rangle - nc\left\langle \left(\beta_\perp D_\perp + \beta_\parallel D_\wedge\right) \tfrac{\partial}{\partial p_\perp} \ln f + \left(\beta_\perp D_\wedge + \beta_\parallel D_\parallel\right) \tfrac{\partial}{\partial p_\parallel} \ln f \right\rangle, \\ \overrightarrow{F}_d &= -neE_\parallel \widehat{e}_\parallel - n\left\langle D_\perp \tfrac{\partial}{\partial p_\perp} \ln f + D_\wedge \tfrac{\partial}{\partial p_\parallel} \ln f \right\rangle \widehat{e}_\perp - n\left\langle D_\wedge \tfrac{\partial}{\partial p_\perp} \ln f + D_\parallel \tfrac{\partial}{\partial p_\parallel} \ln f \right\rangle \widehat{e}_\parallel \\ &= F_\perp \widehat{e}_\perp + F_\parallel \widehat{e}_\parallel. \end{aligned} \tag{9}$$

In the small gyroradius limit $k_\perp v_\perp / (\Omega_e/\gamma) \ll 1$, where $\Omega_e = eB/m_e$, the diffusion coefficients for the fundamental resonance of electron cyclotron waves are given by

$$\begin{aligned} D_\perp &\cong \pi e^2 \lim_{L\to\infty} \tfrac{1}{(2\pi L)^3} \int_k \tfrac{1}{|v_\parallel|} \delta\left(k_\parallel - \tfrac{\omega - \Omega_e/\gamma}{v_\parallel}\right) \left|\tfrac{E_-}{2}\right|^2 \left(1 - \tfrac{k_\parallel v_\parallel}{\omega}\right)^2 d^3k, \\ D_\wedge &\cong \pi e^2 \lim_{L\to\infty} \tfrac{1}{(2\pi L)^3} \int_k \tfrac{1}{|v_\parallel|} \delta\left(k_\parallel - \tfrac{\omega - \Omega_e/\gamma}{v_\parallel}\right) \left|\tfrac{E_-}{2}\right|^2 \left(\tfrac{k_\parallel v_\perp}{\omega}\right) \left(1 - \tfrac{k_\parallel v_\parallel}{\omega}\right) d^3k, \\ D_\parallel &\cong \pi e^2 \lim_{L\to\infty} \tfrac{1}{(2\pi L)^3} \int_k \tfrac{1}{|v_\parallel|} \delta\left(k_\parallel - \tfrac{\omega - \Omega_e/\gamma}{v_\parallel}\right) \left|\tfrac{E_-}{2}\right|^2 \left(\tfrac{k_\parallel v_\perp}{\omega}\right)^2 d^3k, \end{aligned} \tag{10}$$

where L is the linear scale of the system. Only the resonant field component E_- which rotates with the electrons is taken into account (right-hand circularly polarized wave). The integration along k is performed expressing the electric field spectrum as a function of $k_\parallel$, with the $k_\perp$ chosen to satisfy the wave dispersion relation (note that for nearly perpendicular propagation of electron cyclotron waves, i.e. $k_\parallel \cong 0$, the diffusion is predominantly perpendicular). Normalization is such that the effective perpendicular electric field of the wave squared is

$$\overline{E_\perp^2} = \lim_{L\to\infty} \frac{1}{(2\pi L)^3} \int_k \left|\frac{E_-}{2}\right|^2 d^3k = \lim_{L\to\infty} \frac{1}{(2\pi L)} \int_{-\infty}^{\infty} \left|E_{k_\parallel}\right|^2 dk_\parallel \tag{11}$$

and the parallel wave spectrum is expressed in terms of the effective perpendicular wave amplitude $\sqrt{\overline{E_\perp^2}}$ as

$$\left|E_{k_\parallel}\right|^2 = (2\pi L)\, \overline{E_\perp^2} \frac{c}{\omega} S\left(k_\parallel\right), \text{ with } \int_{-\infty}^{\infty} S\left(k_\parallel\right) dk_\parallel = \frac{\omega}{c}. \tag{12}$$

Taking into account the delta-function factor in the $k_\parallel$ integral the diffusion coefficients can be written in the form

$$\begin{aligned} D_\perp &\cong \tfrac{D_0}{|\beta_\parallel|} \left(\tfrac{\Omega_e}{\gamma\omega}\right)^2 \Delta(\xi), \\ D_\wedge &\cong \left(1 - \tfrac{\Omega_e}{\gamma\omega}\right) \tfrac{\gamma\omega}{\Omega_e} \tfrac{\beta_\perp}{\beta_\parallel} D_\perp = \tfrac{D_0}{|\beta_\parallel|} \left(1 - \tfrac{\Omega_e}{\gamma\omega}\right) \tfrac{\Omega_e}{\gamma\omega} \tfrac{\beta_\perp}{\beta_\parallel} \Delta(\xi), \\ D_\parallel &\cong \left(1 - \tfrac{\Omega_e}{\gamma\omega}\right)^2 \left(\tfrac{\gamma\omega}{\Omega_e}\right)^2 \tfrac{\beta_\perp^2}{\beta_\parallel^2} D_\perp = \tfrac{D_0}{|\beta_\parallel|} \left(1 - \tfrac{\Omega_e}{\gamma\omega}\right)^2 \tfrac{\beta_\perp^2}{\beta_\parallel^2} \Delta(\xi), \end{aligned} \tag{13}$$

where $D_0 = \pi e^2 \overline{E_\perp^2}/\omega$, $\xi = \left[1 - \Omega_e/(\gamma\omega)\right]/\beta_\parallel$ and the dimensionless function $\Delta(\xi)$ is defined by

$$\Delta(\xi) = \int_{-\infty}^{\infty} \delta\left(k_\parallel - \frac{\omega - \Omega_e/\gamma}{v_\parallel}\right) S\left(k_\parallel\right) dk_\parallel = S\left(\frac{\omega}{c}\xi\right), \text{ so that } \int_{-\infty}^{\infty} \Delta(\xi)\, d\xi = 1. \tag{14}$$

For a narrow spectrum centered at the refracting index $n_\parallel$, within a small range $\Delta n_\parallel$, the function $\Delta(\xi)$ has the characteristics of a delta-function with unit area and amplitude proportional to $1/\Delta n_\parallel$ at $\xi = n_\parallel$, i.e. $\Delta(\xi) \sim \delta\left(\xi - n_\parallel\right)$.

The power and force components are given in terms of the quasi-delta function $\Delta(\xi)$ by

$$
\begin{aligned}
P_d &\cong -neE_{\|}c\left\langle\beta_{\|}\right\rangle - \tfrac{nD_0}{m_e}\left(\tfrac{\Omega_e}{\omega}\right)\left\langle\tfrac{\beta_\perp^2}{\left|\beta_{\|}\right|}\left(2m_e^2c^2\tfrac{\partial}{\partial p_\perp^2}\ln f + \tfrac{\xi}{1-\xi\beta_{\|}}\tfrac{m_ec}{\gamma}\tfrac{\partial}{\partial p_{\|}}\ln f\right)\Delta(\xi)\right\rangle, \\
F_\perp &\cong -\tfrac{nD_0}{m_ec}\left(\tfrac{\Omega_e}{\omega}\right)\left\langle\tfrac{\beta_\perp}{\left|\beta_{\|}\right|}\left(1-\xi\beta_{\|}\right)\left(2m_e^2c^2\tfrac{\partial}{\partial p_\perp^2}\ln f + \tfrac{\xi}{1-\xi\beta_{\|}}\tfrac{m_ec}{\gamma}\tfrac{\partial}{\partial p_{\|}}\ln f\right)\Delta(\xi)\right\rangle, \\
F_{\|} &\cong -neE_{\|} - \tfrac{nD_0}{m_ec}\left(\tfrac{\Omega_e}{\omega}\right)\left\langle\tfrac{\beta_\perp^2}{\left|\beta_{\|}\right|}\xi\left(2m_e^2c^2\tfrac{\partial}{\partial p_\perp^2}\ln f + \tfrac{\xi}{1-\xi\beta_{\|}}\tfrac{m_ec}{\gamma}\tfrac{\partial}{\partial p_{\|}}\ln f\right)\Delta(\xi)\right\rangle.
\end{aligned}
\tag{15}
$$

These expressions give the macroscopic manifestations of interaction between the energetic particles and both the inductive electric field and a superluminous parallel phase velocity electron-cyclotron wave in the first harmonic EC regime.

Performing the transformation $\left(\xi,\beta_{\|}\right)\to\left(\pi p_\perp^2,p_{\|}\right)$ defined by

$$
\begin{aligned}
\pi p_\perp^2 &= \pi m_e^2c^2\left[\left(\tfrac{\Omega_e}{\omega}\right)^2\tfrac{1-\beta_{\|}^2}{\left(1-\beta_{\|}\xi\right)^2}-1\right], \\
p_{\|} &= m_ec\tfrac{\Omega_e}{\omega}\tfrac{\beta_{\|}}{1-\beta_{\|}\xi}, \\
|J| &= 2\pi m_e^3c^3\left(\tfrac{\Omega_e}{\omega}\right)^3\tfrac{\left|\beta_{\|}\right|}{\left(1-\beta_{\|}\xi\right)^4},
\end{aligned}
\tag{16}
$$

the moments of the Fokker-Planck equation are expressed in the variables $\left(\xi,\beta_{\|}\right)$ by

$$
\left\langle\Phi\left(\overrightarrow{p}\right)\right\rangle = \frac{1}{n}\int\Phi\left(\xi,\beta_{\|}\right)f\left(\xi,\beta_{\|}\right)|J|\,d\xi d\beta_{\|} = \frac{2\pi m_e^3c^3}{n}\left(\frac{\Omega_e}{\omega}\right)^3\int f\left(\xi,\beta_{\|}\right)\frac{\Phi\left(\xi,\beta_{\|}\right)\left|\beta_{\|}\right|}{\left(1-\beta_{\|}\xi\right)^4}d\xi d\beta_{\|}.
\tag{17}
$$

In the limit $\Delta n_{\|}\to 0$ the function $\Delta(\xi)$ becomes a delta function, giving the leading terms in the asymptotic expansions of integrals over the wave spectrum for $\left|n_{\|}\right|<1$:

$$
\left\langle\Phi\left(\xi,\beta_{\|}\right)\Delta(\xi)\right\rangle \sim \frac{2\pi m_e^3c^3}{n}\left(\frac{\Omega_e}{\omega}\right)^3\int f\left(n_{\|},\beta_{\|}\right)\frac{\Phi\left(n_{\|},\beta_{\|}\right)\left|\beta_{\|}\right|}{\left(1-n_{\|}\beta_{\|}\right)^4}d\beta_{\|} = \left\langle\Phi\left(n_{\|},\beta_{\|}\right)\right\rangle_{\xi=n_{\|}}.
\tag{18}
$$

The leading terms in the expressions of the power and force densities are

$$
\begin{aligned}
P_d &\sim -neE_{\|}c\left\langle\beta_{\|}\right\rangle - \tfrac{nD_0}{m_e}\left(\tfrac{\Omega_e}{\omega}\right)\left\langle\tfrac{\beta_\perp^2}{\left|\beta_{\|}\right|}\left(2m_e^2c^2\tfrac{\partial}{\partial p_\perp^2}\ln f + \tfrac{n_{\|}}{1-n_{\|}\beta_{\|}}\tfrac{m_ec}{\gamma}\tfrac{\partial}{\partial p_{\|}}\ln f\right)\right\rangle_{\xi=n_{\|}}, \\
&= -neE_{\|}c\left\langle\beta_{\|}\right\rangle + P_{RF}, \\
F_\perp &\sim -\tfrac{nD_0}{m_ec}\left(\tfrac{\Omega_e}{\omega}\right)\left\langle\tfrac{\beta_\perp}{\left|\beta_{\|}\right|}\left(1-n_{\|}\beta_{\|}\right)\left(2m_e^2c^2\tfrac{\partial}{\partial p_\perp^2}\ln f + \tfrac{n_{\|}}{1-n_{\|}\beta_{\|}}\tfrac{m_ec}{\gamma}\tfrac{\partial}{\partial p_{\|}}\ln f\right)\right\rangle_{\xi=n_{\|}}, \\
F_{\|} &\sim -neE_{\|} + n_{\|}\tfrac{P_{RF}}{c},
\end{aligned}
\tag{19}
$$

where

$$
\beta_\perp^2 = 1-\beta_{\|}^2-\left(\frac{\omega}{\Omega_e}\right)^2\left(1-\xi\beta_{\|}\right)^2,\ \gamma=\frac{\Omega_e}{\omega}\frac{1}{1-\xi\beta_{\|}}.
\tag{20}
$$

4. Distribution function of a magnetized relativistic electron beam

According to the basic plan of this work, a suitable form of the distribution function for the energetic electrons is used to evaluate moments of the Fokker-Planck equation, giving the macroscopic rates of change of density, momentum and energy. In this sense, the simplest approximation is given by a nearly mono-energetic beam. However, besides being unrealistic, a mono-energetic distribution of resonant electrons has ill-defined gradients, which are necessary in the evaluation of the RF diffusion coefficients. To circumvent this difficulty, a fully relativistic bi-Maxwellian representation of the energetic electrons is implemented. This treatment includes a possible loss-cone in the distribution as shown in the following.

The distribution function of a weakly collisional, strongly magnetized relativistic electron beam is

$$f=\frac{1}{h^3}\left(\frac{p_\perp^2}{2m_eT}\right)^\ell\exp\left(\frac{\mu_e m_e}{T_\parallel}-\frac{\gamma m_ec^2-c\left\langle\beta_\parallel\right\rangle p_\parallel}{T_\parallel}-\frac{p_\perp^2}{2m_eT}\right)\tag{21}$$

where h^3 is the unit volume in phase-space, μ_e is the specific chemical potential of the electrons, $\left\langle\beta_\parallel\right\rangle$ is the average velocity in the direction of the magnetic field $\overrightarrow{B}=B\widehat{e}_\parallel$, and $T_\parallel$ defines the parallel temperature in energy units ($T_\parallel<m_ec^2$ to avoid excessive pair production). The temperature T determines the degree of magnetization and $\ell=0,1,2\ldots$ is the loss cone parameter. The constant h is fixed by the normalization

$$n=\int fd^3p=2\pi\int_0^\infty\int_{-\infty}^\infty fp_\perp dp_\perp dp_\parallel=n\left(\ell,\mu_e,\left\langle\beta_\parallel\right\rangle,T_\parallel,T\right).\tag{22}$$

Using the number density n as a primary thermodynamic potential, the momentum density, energy density and stress dyadic are given, respectively, by

$$\begin{aligned}\overrightarrow{G}&=n\left\langle\overrightarrow{p}\right\rangle=\tfrac{T_\parallel}{c}\tfrac{\partial n}{\partial\left\langle\beta_\parallel\right\rangle}\widehat{e}_\parallel=G_\parallel\widehat{e}_\parallel,\\ U&=n\left\langle\gamma m_ec^2\right\rangle=nm_e\mu_e+T_\parallel^2\tfrac{\partial n}{\partial T_\parallel}+c\left\langle\beta_\parallel\right\rangle G_\parallel,\\ \overline{\overline{T}}&=n\left\langle\overrightarrow{v}\,\overrightarrow{p}\right\rangle=nT_\parallel\left[\left(1-\tfrac{T}{n}\tfrac{\partial n}{\partial T}\right)\overline{\overline{I}}_\perp+\widehat{e}_\parallel\widehat{e}_\parallel\right]+c\left\langle\beta_\parallel\right\rangle G_\parallel\widehat{e}_\parallel\widehat{e}_\parallel,\end{aligned}\tag{23}$$

where $\langle\Phi\rangle=n^{-1}\int f\Phi d^3p$, $\overline{\overline{I}}_\perp=\overline{\overline{I}}-\widehat{e}_\parallel\widehat{e}_\parallel$, and the magnetization is given by

$$\overrightarrow{M}=n\left\langle\overrightarrow{\mu}\right\rangle=-n\left\langle\frac{p_\perp^2}{2m_eB}\right\rangle\widehat{e}_\parallel=-\frac{T^2}{B}\frac{\partial n}{\partial T}\widehat{e}_\parallel-\frac{\ell nT}{B}\widehat{e}_\parallel=-\frac{nT}{B}\left(\frac{T}{n}\frac{\partial n}{\partial T}+\ell\right)\widehat{e}_\parallel=M\widehat{e}_\parallel,\tag{24}$$

where μ is the average magnetic moment of the beam electrons. The momentum density, energy density and stress dyadic constitute the energy-momentum tensor in contravariant form, which is written in the laboratory and rest frames of the fluid, respectively, as

$$T^{\mu\nu}=\begin{pmatrix}U & c\overrightarrow{G}\\ c\overrightarrow{G} & \overline{\overline{T}}\end{pmatrix}\text{ and }\overset{0}{T}{}^{\mu\nu}=\begin{pmatrix}\overset{0}{U} & \overrightarrow{0}\\ \overrightarrow{0} & \overset{0}{\overline{\overline{T}}}\end{pmatrix},\tag{25}$$

where

$$\begin{aligned}\overset{0}{U}&=\overset{0}{n}m_e\overset{0}{\mu}_e+\overset{0}{T}{}_\parallel^2\frac{\partial\overset{0}{n}}{\partial\overset{0}{T}_\parallel},\\ \overset{0}{\overline{\overline{T}}}&=\overset{0}{n}\overset{0}{T}_\parallel\left[\left(1-\frac{\overset{0}{T}}{\overset{0}{n}}\frac{\partial\overset{0}{n}}{\partial\overset{0}{T}}\right)\overline{\overline{I}}_\perp+\widehat{e}_\parallel\widehat{e}_\parallel\right],\end{aligned}\tag{26}$$

with the Lorentz transformation properties

$$\overset{0}{n}=n\sqrt{1-\left\langle\beta_\parallel\right\rangle^2},\ \overset{0}{\mu}_e=\frac{\mu_e}{\sqrt{1-\left\langle\beta_\parallel\right\rangle^2}},\ \overset{0}{T}_\parallel=\frac{T_\parallel}{\sqrt{1-\left\langle\beta_\parallel\right\rangle^2}}\text{ and }\overset{0}{T}=T.\tag{27}$$

The quantities $\overrightarrow{G}$, U and $\overline{\overline{T}}$ in the laboratory frame are simply calculated in terms of the rest frame quantities applying a boost $c\left\langle\beta_\parallel\right\rangle$ to the energy-momentum tensor $\overset{0}{T}{}^{\mu\nu}$.

Introducing spherical coordinates $(p,\,\vartheta,\,\varphi)$ so that $p_\perp=p\sin\vartheta$, $p_\parallel=p\cos\vartheta$ and $d^3p=p^2\sin\vartheta\,dp\,d\vartheta\,d\varphi$, changing variables to $\eta=\cos\vartheta$ and $\gamma=\sqrt{1+p^2/\left(m_e^2c^2\right)}$, and moving to the rest frame the normalization condition yields

$$h^3=\frac{4\pi m_e^3c^3}{\overset{0}{n}}\exp\left(\frac{\overset{0}{\mu}_em_e}{\overset{0}{T}_\parallel}\right)\left(\frac{\overset{0}{T}_\parallel}{m_ec^2}\right)\left(\frac{\overset{0}{T}_\parallel}{\overset{0}{T}}\right)^\ell\frac{\Gamma\left(\ell+1/2\right)}{\sqrt{\pi}}\sum_{k=0}^\infty\left(-\frac{\overset{0}{T}_\parallel}{\overset{0}{T}}\right)^kK_{\ell+k+2}\left(\frac{m_ec^2}{\overset{0}{T}_\parallel}\right).\tag{28}$$

Hence, the loss cone distribution function of a magnetized relativistic electron beam can be written in the laboratory frame as

$$f = \frac{1}{h^3}\left(\frac{p_\perp^2}{2m_e\overset{0}{T}}\right)^{\ell} \exp\left(\frac{\overset{0}{\mu}_e m_e}{\overset{0}{T}_\parallel} - \frac{\gamma m_e c^2 - c\left\langle\beta_\parallel\right\rangle p_\parallel}{\overset{0}{T}_\parallel\sqrt{1-\left\langle\beta_\parallel\right\rangle^2}} - \frac{p_\perp^2}{2m_e\overset{0}{T}}\right). \tag{29}$$

Taking $\ell = 0$, $\overset{0}{T} \rightarrow \infty$ and $\left\langle\beta_\parallel\right\rangle = 0$ the Jüttner distribution for an isotropic relativistic electron gas is recovered. Using the distribution function in the above form, the thermodynamic quantities and equations of state pertaining to an anisotropic relativistic electron beam are evaluated. In particular, the Chew-Goldberger-Low double adiabatic equations are recovered in the non-relativistic limit.

The average value of any beam momentum-dependent quantity $\Phi\left(p_\parallel, p_\perp\right)$ with azimuthal symmetry is given in terms of the above distribution function by

$$\begin{aligned}\left\langle\Phi\left(p_\parallel, p_\perp\right)\right\rangle &= \sqrt{1-\left\langle\beta_\parallel\right\rangle^2}\left[\frac{4\pi m_e^3 c^3}{\overset{0}{n} h^3}\exp\left(\frac{\overset{0}{\mu}_e m_e}{\overset{0}{T}_\parallel}\right)\right]\left(\frac{m_e c^2}{2\overset{0}{T}}\right)^{\ell}\tfrac{1}{2}\int_1^\infty d\gamma\,\gamma\left(\gamma^2-1\right)^{l+1/2} \\ &\times \int_{-1}^{+1} d\eta\left(1-\eta^2\right)^{\ell}\exp\left(-\frac{m_e c^2\left(\gamma-\left\langle\beta_\parallel\right\rangle\sqrt{\gamma^2-1}\,\eta\right)}{\overset{0}{T}_\parallel\sqrt{1-\left\langle\beta_\parallel\right\rangle^2}} - \frac{m_e c^2\left(\gamma^2-1\right)\left(1-\eta^2\right)}{2\overset{0}{T}}\right)\Phi\left(\gamma,\eta\right).\end{aligned} \tag{30}$$

Using this expression one evaluates average quantities such as the energy loss by collisions, the parallel momentum loss by collisions, the perpendicular velocity and the perpendicular velocity squared, i.e.

$$\left\langle\frac{\gamma}{\left(\gamma^2-1\right)^{1/2}}\right\rangle,\ \left\langle\frac{\left(1+Z+\gamma\right)\gamma^2}{\left(\gamma^2-1\right)^{3/2}}\beta_\parallel\right\rangle,\ \left\langle\beta_\perp\right\rangle \text{ and } \left\langle\beta_\perp^2\right\rangle, \tag{31}$$

with the replacements

$$\beta_\parallel = \left(\frac{\sqrt{\gamma^2-1}}{\gamma}\right)\eta \text{ and } \beta_\perp = \left(\frac{\sqrt{\gamma^2-1}}{\gamma}\right)\sqrt{1-\eta^2}. \tag{32}$$

Taking the derivatives of the distribution function

$$\begin{aligned}\frac{\partial}{\partial p_\perp^2}\ln f &= \frac{\ell}{p_\perp^2} - \frac{1}{2m_e}\left(\frac{1}{\overset{0}{T}} + \frac{1}{\gamma\overset{0}{T}_\parallel\sqrt{1-\left\langle\beta_\parallel\right\rangle^2}}\right), \\ \frac{\partial}{\partial p_\parallel}\ln f &= -\frac{c}{\overset{0}{T}_\parallel\sqrt{1-\left\langle\beta_\parallel\right\rangle^2}}\left(\beta_\parallel - \left\langle\beta_\parallel\right\rangle\right),\end{aligned} \tag{33}$$

and expressing them in the $(\xi,\ \beta_\parallel)$ variables one finds

$$\begin{aligned}P_{RF} &\sim \frac{D_0}{m_e}\left(\frac{\Omega_e}{\omega}\right)^4\left(\frac{m_e c^2}{\overset{0}{T}} + \frac{\omega}{\Omega_e}\frac{m_e c^2\left(1-n_\parallel\left\langle\beta_\parallel\right\rangle\right)}{\overset{0}{T}_\parallel\sqrt{1-\left\langle\beta_\parallel\right\rangle^2}}\right)2\pi m_e^3 c^3\int_{\beta_-}^{\beta_+}\frac{f\left(n_\parallel,\beta_\parallel\right)\beta_\perp^2}{\left(1-n_\parallel\beta_\parallel\right)^4}d\beta_\parallel \\ &\quad -2\ell\frac{D_0}{m_e}\left(\frac{\Omega_e}{\omega}\right)^2 2\pi m_e^3 c^3\int_{\beta_-}^{\beta_+}\frac{f_e\left(n_\parallel,\beta_\parallel\right)}{\left(1-n_\parallel\beta_\parallel\right)^2}d\beta_\parallel, \\ F_\perp &\sim \frac{D_0}{m_e c}\left(\frac{\Omega_e}{\omega}\right)^4\left(\frac{m_e c^2}{\overset{0}{T}} + \frac{\omega}{\Omega_e}\frac{m_e c^2\left(1-n_\parallel\left\langle\beta_\parallel\right\rangle\right)}{\overset{0}{T}_\parallel\sqrt{1-\left\langle\beta_\parallel\right\rangle^2}}\right)2\pi m_e^3 c^3\int_{\beta_-}^{\beta_+}\frac{f\left(n_\parallel,\beta_\parallel\right)\beta_\perp}{\left(1-n_\parallel\beta_\parallel\right)^3}d\beta_\parallel \\ &\quad -2\ell\frac{D_0}{m_e c}\left(\frac{\Omega_e}{\omega}\right)^2 2\pi m_e^3 c^3\int_{\beta_-}^{\beta_+}\frac{f_e\left(n_\parallel,\beta_\parallel\right)}{\left(1-n_\parallel\beta_\parallel\right)\beta_\perp}d\beta_\parallel,\end{aligned} \tag{34}$$

where the integration limits are defined by the two roots of $\beta_\perp^2$:

$$\beta_\pm = \frac{n_\parallel \pm \left(\Omega_e/\omega\right)\sqrt{\left(\Omega_e/\omega\right)^2 + n_\parallel^2 - 1}}{\left(\Omega_e/\omega\right)^2 + n_\parallel^2}, \tag{35}$$

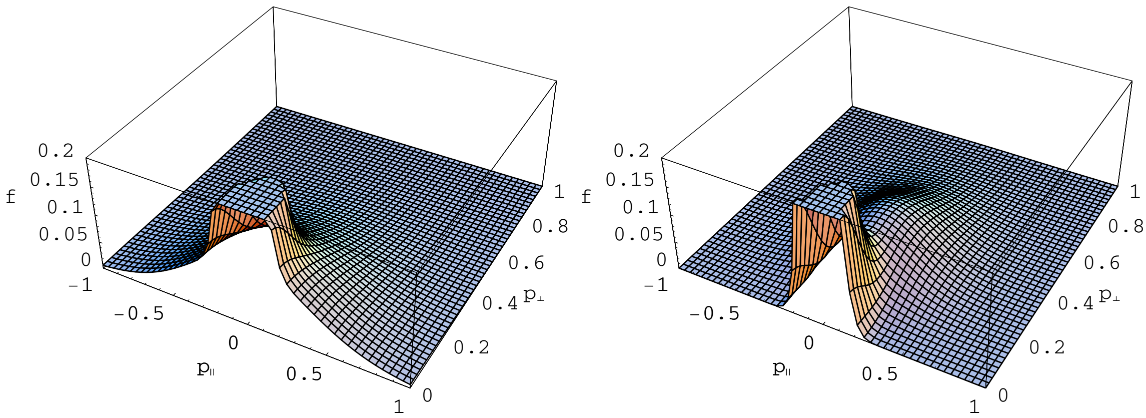

Figure 1: *Electron distribution function truncated at 0.2 of its maximum value for TCV ($R_0 = 0.88\,m$, $a = 0.25\,m$, $\kappa = 1.5$, $B = 1.43\,T$) fully ECCD driven discharges. The momentum components are normalized to $m_e c$ and the background plasma has density $n_e = 10^{19}\,m^3$ and temperature $T_e = 3.5\,keV$. The plots on the left and right-hand sides correspond to toroidal launching angles 15^o and 25^o, respectively.*

5. Steady-state ECCD discharge

The energy and parallel momentum conservation equations constitute a set of two equations for the three macroscopic quantities $\langle\beta_\parallel\rangle$, $T_\parallel$ and T that characterize the distribution function of the energetic current-carrying electrons. This set of equations is complemented by the general relationship between the macroscopic power and force per unit volume acting on the energetic particles, $P_d = c\vec{F}_d \cdot \left\langle\vec{\beta}\right\rangle$, which gives the average perpendicular velocity of the particles induced by RF waves

$$\langle\beta_\perp\rangle \sim \frac{\left(1 - n_\parallel \langle\beta_\parallel\rangle\right) P_{RF}}{cF_\perp}. \tag{36}$$

The expression of the parallel force density, $F_\parallel \sim -neE_\parallel + n_\parallel P_{RF}/c$, clearly shows that the RF driven current vanishes when $n_\parallel \to 0$. For a fully RF driven tokamak plasma the inductive electric field is put equal to zero. In the steady-state a global figure of merit is defined by

$$\zeta_{RF} = \frac{I_{RF}}{W_{RF}} = \frac{e\langle\beta_\parallel\rangle A_p}{m_e c\nu_c P_{RF} V_p}, \tag{37}$$

where A_p is the area of the poloidal cross-section and V_p the volume of the plasma. Using the present fluid model a series of equilibrium solutions was determined for the DIII-D and TCV tokamak operating conditions [1, 2, 3, 6]. In the case of TCV, a theoretical scan in $n_\parallel$ was made for fixed input power $W_{RF} = nm_e c^2 \nu_c P_{RF} V_p \Delta A_p / A_p = 1.5\,\text{MW}$ and fixed driven current $I_{RF} = nec\langle\beta_\parallel\rangle \Delta A_p = 105\,\text{kA}$ (that is, fixed ζ_{RF}), where ΔA_p is the area of the poloidal cross-section over which the RF power is deposited. Two equilibrium solutions for TCV are shown in Fig. 1. The plot on the left-hand side corresponds to a toroidal launching angle $\varphi_T = 15^o$ ($n_\parallel = \sin\varphi_T$) and the plot on the right-hand side to $\varphi_T = 25^o$. On the left-hand side plot one clearly sees the contribution of the anisotropic beam of high-energy electrons with $\ell = 0$ to the tail of the background isotropic plasma, while on the right-hand side the beam assumes a loss-cone distribution with $\ell = 1$. The transition between these two solutions occurs for an angle $\varphi_T \lesssim 20^o$, when the beam is practically isotropic, i.e. demagnetized ($\overset{0}{T} \gg 1$).

6. Instabilities in a plasma with RF-driven current

The equilibrium solutions described in the previous section were examined for various kinds of instabilities in plasmas traversed by electron beams. The analysis shows that, because of the large parallel temperature and relatively small streaming velocity of the beam, the electrostatic beam-plasma instabilities are inhibited. However, the electromagnetic (Weibel) instabilities can be excited because of the beam anisotropy. In particular, the right-hand circularly polarized wave (whistler mode) propagating in the background plasma becomes unstable in the presence of the beam with a loss-cone configuration. Assuming small growth rates in the long wavelength approximation, the real part $\omega_k < \Omega_e$ of the frequency satisfies the whistler mode dispersion relation allowing to evaluate the growth or damping rate γ_k:

$$k_\parallel c \cong \omega_k \left(1 + \frac{\omega_e^2}{\omega_k(\Omega_e - \omega_k)}\right)^{1/2},$$
$$\gamma_k \cong \sqrt{\pi}\,\omega_k \left(\frac{\omega_e^2}{2k_\parallel^2 c^2}\right)\left(1 + \frac{\omega_e^2 \omega_k (2\omega_k - \Omega_e)}{2k_\parallel^2 c^2 (\Omega_e - \omega_k)^2}\right)^{-1} \left\{ -\left(\frac{\omega_k}{k_\parallel \sqrt{2T_e/m_e}}\right) \exp\left(-\frac{(\Omega_e - \omega_k)^2}{2k_\parallel^2 T_e/m_e}\right) \right.$$
$$\left. + \frac{n}{n_e}\left(\frac{\omega_k - k_\parallel c\langle\beta_\parallel\rangle}{k_\parallel\sqrt{2T_\parallel/m_e}}\right)\left[\left(\frac{(\ell+1)!\,T_\perp}{\ell!\,T_\parallel} - 1\right)\left(\frac{\Omega_e - (\omega_k - k_\parallel c\langle\beta_\parallel\rangle)}{\omega_k - k_\parallel c\langle\beta_\parallel\rangle}\right) - 1\right] \exp\left(-\frac{\left[\Omega_e - (\omega_k - k_\parallel c\langle\beta_\parallel\rangle)\right]^2}{2k_\parallel^2 T_\parallel/m_e}\right)\right\}, \tag{38}$$

where the effective perpendicular temperature is $T_\perp \sim \left(1/T_\parallel + 1/T\right)^{-1}$ in the non-relativistic limit (a precise formulation of the dispersion relation using the fully relativistic distribution function is still in development). For small values of $n_\parallel$ ($\varphi_T \lesssim 20^o$) one has $\ell = 0$, $T_\perp < T_\parallel$ and damping of the whistler mode. Beyond the transition to a loss-cone configuration ($\varphi_T \gtrsim 20^o$) one has $\ell = 1$, $T_\perp \sim T_\parallel$ and a growth rate which is proportional to the ratio n/n_e of beam to plasma densities (electron Landau damping is negligible for φ_T somewhat larger than 20^o in TCV). The growth is maintained above marginal instability on a region defined by the condition

$$\omega_k - k_\parallel c\,\langle\beta_\parallel\rangle < \left(1 - \frac{\ell!\,T_\parallel}{(\ell+1)!\,T_\perp}\right)\Omega_e. \tag{39}$$

In this weakly unstable situation a group of particles drifting in opposite direction to the phase velocity of the wave has Doppler shifted velocities that allow cyclotron resonance with the wave. The wave growth is associated to pitch angle scattering of the beam electrons, transferring energy from the perpendicular to the parallel direction. It must be pointed out that the equilibrium solutions, although idealized, are determined in the absence of radial transport and the whistler instability indicates an additional loss mechanism of the beam particles mainly by pitch angle scattering, thus reducing the ECCD efficiency.

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References

[1] O. Sauter et al., *Phys. Rev. Lett.* **84**, 3322-5 (2000).

[2] R.W. Harvey, O. Sauter, R. Prater and P. Nikkola, *Phys. Rev. Lett.* **88**, 2050001-1-4 (2002).

[3] C.C. Petty et al., *Nucl. Fusion* **42**, 1366-1375 (2002).

[4] R.W. Harvey and M.G. McCoy, “The CQL3D Fokker-Planck code” *Proc. IAEA Technical Committee Meeting on Advances in Simulation and Modeling in Thermonuclear Plasmas* (Montreal, 1992) (Vienna: IAEA) 498-536 (1993).

[5] G. Giruzzi, *Plasma Phys. Control. Fusion* **35**, A123-39 (1993).

[6] S. Coda et al., *Nucl. Fusion* **43**, 1361-1370 (2003).

[7] K. Hizanidis and A. Bers, *Phys. Fluids* **27**, 2669-2675 (1984).

[8] N.J. Fisch, *Phys. Rev. Lett.* **41**, 873-6 (1978); N.J. Fisch and A.H. Boozer, *Phys. Rev. Lett.* **45**, 720-2 (1980); N.J. Fisch, *Phys. Rev. A* **24**, 3245-8 (1981).

[9] P. Penfield, Jr. and H.A. Haus, *Electrodynamics of Moving Media* (MIT, Cambridge) p. 13 (1967).