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THE LENGTH OF THE INSTABILITY INTERVALS OF THE MODULATED FREQUENCY EQUATION

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1. Introduction

Lord Rayleigh seems to have been the first one to have given a theoretical explanation of the oscillations of a system in which the stiffness parameter is periodically varied. In 1887 published a discussion, [15], of some of the types of oscillations of a string whose tension is periodically altered. His mention at that time of the corresponding situation in an electric circuit anticipated the interest to be given to this kind of vibratory motion some thirty years later, in the fields of radio communication and electro-acoustics. In 1922, J.R. Carson [7], discussed, from the point of view of radiotelephony an actual electric circuit with inductance and a sinusoidally varying capacitance. The response of such a circuit is governed by the two-parameters, second order, ordinary differential equation with periodic coefficient,

$$(1+\varepsilon \cos 2t)x'' + \lambda x = 0 , \qquad (1.1)$$

where ε and λ are real.

The purpose of the present work is to show that, for $|\varepsilon| < 1$, the λ axis consists of intervals in which all solutions of (1.1) are bounded, and intervals in which (1.1) posseses at least one unbounded solution. Furthermore, our main goal is to determine the lengths of the intervals where unbounded solutions exist. These intervals will be called instability intervals. Our main result is contained in the following theorem:

<u>Theorem 1</u>: The length L_n of the n-th instability interval of the modulated frequency equation (1.1) is given by:

$$L_n = 0 \quad \text{if n is even} \tag{1.2}$$

$$L_n = \frac{1}{8^{n-1}}\left[\frac{n!!}{(n-1)!!}\right]^2 \varepsilon^n + o(\varepsilon^n) \quad \text{if n is odd.}$$

$$(m!! = \prod_{0 \le 2j < m} (m-2j)) \tag{1.3}$$

Equation (1.2) is a known result, see for example Magnus and Winkler, [13]. Included here for a sake of completeness, it naturally appears in the process of proving (1.3).

2. The Modulated Frequency Equation

Let us consider an L-C electrical circuit consisting of a constant inductance L and a time-varying capacity $C(\tau)$. Let i be the current and q the charge of the condenser, then, if no amount of q is removed from the circuit at any time, we have

$$\frac{d}{d\tau}(Li) + \frac{q}{C(\tau)} = 0 \tag{2.1}$$

Since

$$i = \frac{d}{d\tau} q \ ,$$

equation (2.1) is transformed into

$$LC(\tau) \frac{d^2}{d\tau^2} q + q = 0 \tag{2.2}$$

Assuming

$$C(\tau) = C_o + \Delta C.\cos(p\tau) \;, \qquad (2.3)$$

which, for example, represents the conditions when a sinusoidal note of frequency $p/2\Pi$ is sung in front a condenser transmiter, one obtains the <u>modulated frequency equation</u>

$$(1+\varepsilon \cos 2t)x'' + \lambda x = 0 \qquad (()' = \frac{d}{dt}) \qquad (2.4)$$

from (2.2), substituting $C(\tau)$ by (2.3), and setting

$$\varepsilon = \frac{\Delta C}{C_o} \;;\; p\tau = 2t \;;\; x(t) = q(2t/p) \;;\; \omega_o^2 = \frac{1}{LC_o} \;;$$

$$\lambda = (2\omega_o/p)^2 \;.$$

We shall assume in what follows that $|\varepsilon| < 1$. Furthermore, since there is no loss of generality, we shall only consider $\varepsilon > 0$, otherwise, one can change t by $t+\Pi/2$ obtaining the same equation with positive ε.

Although equation (2.4) is not singular, division by $1+\varepsilon \cos 2t$ does not lead to a typical Hill's equation, furthermore, the usual expansion of $1/(1+\varepsilon \cos 2t)$ in power series by long division and subsequent first order approximation which yields Mathieu's equation conduces to erroneous results as it is shown by (1.2). However, as for the Hill's equation, an oscillation theorem, (to be proved somewhere else), is valid, that is, for $0 \leq \varepsilon < 1$ there exist two monotonically increasing sequences of real numbers.

$$\lambda_o, \lambda_1, \lambda_2, \ldots$$

and

$$\lambda_1', \lambda_2', \lambda_3', \ldots$$

such that the modulated frequency equation (2.4) posseses a solution of period Π if and only if $\lambda = \lambda_n$, n=0,1,2,... and a solution of period 2Π if and only if $\lambda = \lambda'_n$, n=1,2,3,... The λ_n and λ'_n satisfy

$$0 = \lambda_o < \lambda'_1 \leq \lambda'_2 \leq \lambda_1 \leq \lambda_2 < \lambda'_3 \leq \lambda'_4 \ldots$$

and the relations

$$\lim_{n\to\infty} \lambda_n^{-1} = \lim_{n\to\infty} (\lambda'_n)^{-1} = 0 \ .$$

The solutions are bounded in the intervals

$$(\lambda_o, \lambda'_1),\ (\lambda'_2, \lambda_1),\ (\lambda_2, \lambda'_3),\ \text{etc}$$

and there is at least one unbounded in the complement.

Thus, in order to determine the lengths of the instability intervals we only need to determine the values $\lambda = \lambda(\varepsilon)$ for which (2.4) has a periodic solution. To this end, we apply a combined method of power series in ε and Fourier series in t.

3. Solution of the Modulated Frequency Equation

It follows from the eveness of cos 2t and the linearity of the equation, that when (2.4) has a periodic solution of any period, Π or 2Π, it also has an even and an odd solution of the same period, therefore, we may consider only such solutions in determining the λ_n and λ'_n.

We now proceed to solve (2.4) assuming

$$x(t) = \sum_{j=0}^{\infty} x_j(t)\varepsilon^j \tag{3.1}$$

$$\lambda(\varepsilon) = \sum_{j=0}^{\infty} \alpha_j \varepsilon^j \tag{3.2}$$

Substituting x and λ in (2.4) by (3.1) and (3.2) respectively, we have, upon collecting powers of ε and equating to zero their coefficients:

$$x_o'' + \alpha_o x_o = 0 \tag{3.3}$$

$$x_j'' + \alpha_o x_j = -\cos 2t\, x_{j-1}(t) - \sum_{i=1}^{j} \alpha_i x_{j-i}(t)$$

$$1 \le j \tag{3.4}$$

Equation (3.3) has the even and odd periodic solutions

$$\cos(\alpha_o^{1/2} t),\ \sin(\alpha_o^{1/2} t)\ .$$

These solutions have period 2Π if $\alpha_o^{1/2}$ is an integer n, and period Π if, in addition, such n is even. Thus we have, with the superscripts + and - denoting even and odd respectively

$$\alpha_o^{\pm} = n^2 \qquad n=1,2,... \tag{3.5}$$

$$x_o^{+}(t) = \cos nt \tag{3.6}$$

$$x_o^{-}(t) = \sin nt \tag{3.7}$$

In solving (3.4) for $j \geq 1$ one obtains, for each j, a solution which is not unique as it involves an arbitrary solution of the homogeneous equation (3.3). To make them unique, we require that x(t) satisfy the condition

$$\frac{1}{\Pi}\int_0^{2\Pi} x(t)x_o(t)dt = 1 \qquad (3.8)$$

We now expand x_j^+ and x_j^- in the Fourier Series

$$x_j^+(t) = \sum_{k=0}^{\infty} x_{jk}^+ \cos kt \qquad (3.9)$$

$$x_j^-(t) = \sum_{k=1}^{\infty} x_{jk}^- \sin kt \qquad (3.10)$$

Inserting (3.9), (3.10) in (3.4) and equating coefficients one obtains

$$n^2 x_{jo}^+ + \sum_{i=1}^{j} \alpha_i^+ x_{j-i,o}^+ = 2x_{j-1,2}^+ \qquad (3.11)$$

$$(n^2-1)x_{j1}^{\pm} + \sum_{i=1}^{j} \alpha_i^{\pm} x_{j-i,1}^{\pm} = \frac{1}{2}[\pm x_{j-1,1}^{\pm} + 9x_{j-1,3}^{\pm}] \qquad (3.12)$$

$$(n^2-k^2)x_{jk}^{\pm} + \sum_{i=1}^{j} \alpha_i^{\pm} x_{j-i,k}^{\pm} = \frac{1}{2}[(k-2)^2 x_{j-1,k-2}^{\pm} + (k+2)^2 x_{j-1,k+2}^{\pm}]. \qquad (3.13)$$

where $\alpha_j^{\pm}$ now denotes the coefficient α_j in (3.2) corresponding to an even or an odd solution respectively.

It follows from (3.1), (3.6)-(3.10) that

$$x_{jn}^{\pm} = 0 \qquad j > 0 \qquad (3.14)$$

and

$$x_{ok}^{\pm} = \delta_{kn} . \qquad (3.15)$$

Equations (3.5) and (3.15) yield the values of all the unknowns when $j = 0$. Assuming one also has $\alpha_i^\pm$ and $x_{ik}^\pm$ for i up to some j-1 and every k, setting k = m in (3.13) produces

$$\alpha_j^\pm = \frac{1}{2}[(n-2)^2 x_{j-1,n-2}^\pm + (n+2)^2 x_{j-1,n+2}^\pm] \qquad (3.16)$$

which is then a known quantity and thus (3.13), (or (3.12) if k=1), can be used to determine $x_{jk}^\pm$.

Once the $\alpha_j^\pm$'s have been calculated, the length L_n of the n-th instability interval can be found, i.e.

$$L_n(\varepsilon) = \lfloor \lambda^+(\varepsilon) - \lambda^-(\varepsilon) \rfloor = \left[\sum_{j=0}^{\infty} (\alpha_j^+(\varepsilon) - \alpha_j^-(\varepsilon)) \varepsilon^j \right] \qquad (3.17)$$

4. Lemmata

The following Lemmas give the necessary elements to prove theorem 1 from (3.17). All this lemmas can be proved by induction. Their proofs, omitted here for a sake of brevity, will appear somewhere else.

Lemma 1: $x_{jk}^\pm = 0$ when n and k are of different parity.

Lemma 2: $x_{jk}^\pm = 0$ if $k < n-2j$ or $k > n+2j$

Lemma 3: If n is even, then $x_{jk}^+ = x_{jk}^-$ for all $j \geq 0$ and $k > 0$. Also $\alpha_j^+ = \alpha_j^-$ for all j.

Lemma 4: For even values of n, $x_{jk}^+ = x_{jk}^- = 0$ if j is even and $k = n \pm 2 \pm 4\ell$ for some ℓ and $x_{jk}^+ = x_{jk}^- = 0$, $\alpha_j^+ = \alpha_j^- = 0$ if j is odd and $k = n \pm 4\ell$ for some ℓ.

We shall, from now on, consider the problem only in the case n odd. Although theorem 1 holds when n=1,3,5, the next two lemmas would require some modifications if these cases were included, thus we give the coefficients $\alpha_j^{\pm}$ for n=1,3,5 $0 \le j \le n$ in Table I and proceed to consider $n \ge 7$.

Lemma 5: Let $j_o = (n-1)/2$. Then:

for $1 \le j \le j_o$,

a) $\alpha_j^{\pm} = 0$ if j is odd, and $x_{jk}^{\pm} = 0$ if, in addition, $k=n\pm 4\ell$ for some ℓ.

b) $x_{jk}^{\pm} = 0$ if j is even and $k=n\pm 2\pm 4\ell$ for some ℓ.

For $j_o < j \le n-1$, a) and b) are valid provided $k \ge 2(j+1)-n$ is required.

Lemma 6: $x_{jk}^{+} = x_{jk}^{-}$ for all k if $j \le j_o$ and for $k \ge 2(j+1)-n$ if $j_o < j$.

Corollary: $\alpha_j^{+} = \alpha_j^{-}$ for $0 \le j \le n-1$.

5. Proof of Theorem 1

According to lemmas 1-3, the instability intervals between $\lambda^{+}(\varepsilon)$ and $\lambda^{-}(\varepsilon)$ with $\lambda^{+}(0) = \lambda^{-}(0) = n$ desappear when n is even, hence, $L_n = 0$ and the first part of the theorem is proved. Lemmas 5 and 6 and the corollary imply

$$L_n = (\alpha_n^{+} - \alpha_n^{-})\varepsilon^n + o(\varepsilon^n)$$

when n is odd. Thus, if $\alpha_n^{+} \neq \alpha_n^{-}$ we have that the first non-vanishing term is of order n.

In order to calculate $\alpha_n^{\pm}$ we apply (3.16) observing that $x_{n-1,n+2}^{\pm} = 0$ because n-1 is even and $n+2 > 2(n-2)-n$

and therefore Lemma 5 holds. Hence,

$$\alpha_n^{\pm} = \frac{1}{2}(n-2)^2 x_{n-1,n-2}^{\pm} . \qquad (4.1)$$

Using (3.13),

$$x_{n-1,n-2}^{\pm} = \frac{1}{n^2-(n-2)^2} \{-\sum_{i=1}^{n-1} \alpha_i^{\pm} x_{n-1-i,n-2}^{\pm} + \frac{1}{2}[(n-4)^2 x_{n-2,n-4}^{\pm} + n^2 x_{n-2,n}^{\pm}]\} . \qquad (4.2)$$

We know that $\alpha_i^{\pm} = 0$ when i is odd; for i even, n-1-i is even and n-2 is a number in the form $n\pm2\pm4\ell$, moreover, $n-2 \geq 2(n-1-i+1)-n$ and lemma 5 can be applied. Condition (3.15) yields $x_{n-2,n}^{\pm} = 0$ on the right hand side of (4.2). Therefore

$$x_{n-1,n-2}^{\pm} = \frac{(n-4)^2}{2[n^2-(n-2)^2]} x_{n-2,n-4}^{\pm}$$

One can obtain, by the same token,

$$x_{n-s,n-2s}^{\pm} = \frac{[n-2(s+1)]^2}{2[n^2-(n-2s)^2]} x_{n-(s+1),n-2(s+1)}^{\pm}$$

$$1 \leq s \leq j_o - 1 . \qquad (4.3)$$

Hence

$$x_{n-1,n-2}^{\pm} = \frac{\prod_{s=1}^{j_o-1} [n-2(s+1)]^2}{2^{j_o-1} \prod_{s=1}^{j_o-1} [n^2-(n-2s)^2]} x_{j_o+1,1}^{\pm} \qquad (4.4)$$

We now use (3.13) and Lemmas 4 and 5 to obtain

$$x^{\pm}_{j_o+1,1} = \pm \frac{1}{2(n^2-1)} x^{\pm}_{j_o,1} \tag{4.5}$$

and

$$x^{\pm}_{j_o-s,1+2s} = \frac{[1+2(s+1)]^2}{2[n^2-(1+2s)^2]} x^{\pm}_{j_o-(s+1),1+2(s+1)}$$

Then

$$x^{\pm}_{j_o,1} = \frac{\prod_{s=0}^{j_o-1}[1+2(s+1)]^2}{2^{j_o}\prod_{s=0}^{j_o-1}[n^2-(1+2s)^2]} \tag{4.6}$$

Using formulas (4.1)-(4.6) backwards, we have

$$\alpha^{\pm}_n = \pm \frac{4}{8^n}\left[\frac{n!!}{(n-1)!!}\right]^2 , \tag{4.7}$$

and then

$$L_n = \frac{1}{8^{n-1}}\left[\frac{n!!}{(n-1)!!}\right]^2 \varepsilon^n + o(\varepsilon^n) \tag{4.8}$$

TABLE I

j \ $\alpha^{\pm}_j$	m=1	m=3	m=5
0	1	9	25
1	±0,5	0	0
2		-3.234375	-9.244792
3		±0.017578	0
4			-2.453543
5			±0.000429

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