NUMERICALLY SOLVING EIGENVALUE EQUATIONS AS MINIMIZATION PROBLEMS

ABSTRACT

The purpose of this paper is to present, through two study cases, an alternative way of numerically solving parameter depending algebraic or differential equations, by using minimization techniques of Operation Research.

INTRODUCTION

Very often, standard methods for solving algebraic or differential equations which depend on a n-dimensional set of parameters, fail to converge or solve the problem at all due to the lack of differentiability of the equations with respect to the parameters. Our proposal here is to define an associated non-negative function which has a minimum at the solution of the original problem, determining then the location of the minimum, i.e., the solution of the original problem, by any one of the minimization techniques of Operations Research.

EXAMPLE 1

This is a very simple example, and it is quite obvious that nobody would attempt to solve it in the way we are about to propose, however, it is its simplicity which will allow us to expose the method.

Let us consider the Initial Boundary Value Problem of finding 2π - periodic solutions of the pendulum equation

(1.1) $x'' + w^2 \sin(x) = 0$

subjected to the initial condition

(1.2) $x'(0)=0$.

It is a well known fact that

(1.3) $w =$

where $a=x(0)$ is the amplitude of the solution and Sn is the elliptic sine (Stoker J.J., 1954).

Whichever the method applied to find the solutions of (1.1),(1.2), it will, in general terms, follow the algorithm:

I .- Choose a pair of values a and w.
II .- Integrate (1.1)
III .- If the function obtained is 2π - periodic the problem has been solved, if it is not, choose another value of a or of w and go to II.

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Our proposal is to go through the steps I-III in a systematic way, without relying on the differentiability of the operator or the initial/boundary conditions. To do that, let us define the function

$$(1.4) \qquad F_a[w] = \left| a - x(2\pi) \right| + \left| x'(2\pi) \right|$$

where $x(t)$ is the function obtained by numerical integration of (1.1) for the given frequency w and initial value a.

Clearly $F_a[w] \geq 0$, and it is equal to zero if and only if $x(t)$ is 2π-periodic, i.e., for fixed a, $F_a[w]$ has a local minimum at the value of w defined by (1.3). We have thus transformed the problem of finding 2π-periodic solutions of (1.1),(1.2) into en equivalent minimization problem. An optimizing algorithm can now be applied to solve the latter, what amounts to systematically search for the value of w for which $F_a[w]=0$, then, for that w, the corresponding $x(t)$ is a solution of the original problem. The example treated here is a one dimensional problem and therefore it is most natural to apply the Golden Section method (Fibonacci sequences) (Press et alli,1986) to determine the minimum of (1.3). Given a and an interval $[w_1,w_2]$ where $w(a)$ can be found, the Golden method will determine points w_3 and w_4 in that interval, evaluate (1.3) at w_j $j=1,2,3,4$, bracketing the minimum between w_1 and w_4 or w_3 and w_2 depending on the relative values of $F_a[w]$. It is important to observe that at each step only two evaluations are necessary because two of the values obtained in the previous step are utilized. Figure 1 shows the graph of the numerical results obtained in this way, and for the sake of comparison, the curve $w = 1. + a^2/16$ obtained by a second order perturbation approximation.

EXAMPLE II

Let us consider the problem of solving the balance equations of a ram-jet (referencia do ram-jet)

(2.1)

(2.2)

(2.3)

(2.4)

(preencher equacoes)

In (2.1)-(2.4) is ....(preencher variaveis com o significado)

It is clear in this case that standard methods would fail, mainly due to the non-linearity of the equations with respect to epsilon. We thus define the function

(2.2) F[...variaveis...]=

which attains its minimum value, zero, at the roots of (2.1)-(2.4). Once again we have transformed the original problem of solving a non-linear system of trascendental equations into a multidimensional minimization problem. We used the Hooke and Jeeves (Novaes A. G., 1978) algorithm to minimize F. Given an hypercube, where the solution must be located, and an initial guess X_0 in it, the algorithm varies one parameter at a time, looking for a value of F smaller than $F[X_0]$. When it is determined, saves it location and varies the next coordinate. If there is no possible improvement then the minimum was attained at X_0, otherwise, a point Y is determined with $F[Y] < F[X_0]$. The algorithm checks then in the direction X_0 --> Y, comparing F[Y] with

F[Z], where Z is a point on the straight line X_0 - Y but farther away from X_0 than Y.

Table 1 shows the solution of (2.1) for the indicated parameters.

CONCLUSIONS

We have just shown that given the eigenvalue problem of finding λ and X such that

(3.1) $f[X,\lambda] = 0$

subjected to conditions

(3.2) $g[X,\lambda] = 0$, (they might be trivial conditions)

if one defines

$$F_X[\lambda] = \| f[X,\lambda] \| + \| g[X,\lambda] \|,$$

then the eigenvalue problem can be solved by minimizing F using the most suitable algorithm for that. This way of solving (3.1),(3.2) is not in general an efficient one when there are other methods available, however, it can be of great help when f or g are not smooth enough or when one wishes to have an estimate for the starting value of an iterative process.

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