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Analytical Space-Periodic Solutions of the Kuramoto-Sivashinsky Equation

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1.- INTRODUCTION

This paper serves a twofold purpose. First we evaluate space-periodic solutions of

$$\frac{\partial\Phi}{\partial t}+\frac{\partial^4\Phi}{\partial x^4}+\frac{\partial^2\Phi}{\partial x^2}+\frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2=0 \qquad (1)$$

using perturbation methods and Fourier decomposition. Second, we employ those solutions to determine progressive-wave solutions of (1), and discuss some of their properties and physical meaning.

Equation (1), known as the one-dimensional Kuramoto-Sivashinsky equation, was obtained by Kuramoto [1]. It is a particular case of the Michelson-Sivashinsky equation[2,3] derived as a model of flame propagation. In this case, Φ is proportional to the dimensionless perturbation of a flame front, and it is in this context that we shall analyze our results.

Michelson[4], in a numerical study, has shown that when (1) is integrated over an interval $-\ell < x < \ell$, ℓ large, with periodic boundary conditions, it has a solution function $\Phi(x,t)$ of the form

$$\Phi(x,t) = -\mu\, t + \phi(x,t)\,,\quad \mu \text{ constant} \qquad (2)$$

Furthermore, for fixed t, $\phi(x,t)$ resembles a quasi-periodic function. In a later paper, Troy[5] proved the existence of steady solutions of (1).

We shall investigate here the case in which $\phi(x,t)$ is actually independent of t. Furthermore, we seek functions $\phi(x)$ periodic with fundamental period $2\pi/k$ for some k.

For fixed, given k,let

$$\mu = k^2\sigma, \qquad z = k\,x, \qquad x\in[0,2\pi/k]$$

$$f(z) = \phi\,(z/k), \qquad z\in[0,2\pi] \qquad (3)$$

Substituting Φ and x in (1) by (2) and (3) we obtain, after simplifying k^2,

$$k^2 f^{iv} + f'' + \frac{1}{2}(f')^2 = \sigma \quad ()' = d/dz \qquad (4)$$

where $f(z+2\pi) = f(z)$.

Let us consider the artificially modified equation

$$k^2 f^{iv} + \lambda f'' + \frac{1}{2}(f')^2 = \sigma \qquad (5)$$

When $\sigma = 0$, $f = f_0 \equiv 0$ is a solution of (5) for all values of λ. It is a straightforward application of the bifurcation theorem[6] to show that a 2π-periodic solution f(z) of (5) bifurcates from f_0 at $\lambda = k^2$. According to the theorem, f(z), as well as the values of σ and λ for which it exists, can be obtained parametrically as $f = f(z,\varepsilon)$, $\lambda=\lambda(\varepsilon)$, $\sigma=\sigma(\varepsilon)$, in terms of a parameter ε, such that $f(z,0) = 0$,

$\lambda(0) = k^2$, $\sigma(0) = 0$. The parameter ε can be defined by

$$\frac{1}{\pi}\int_0^{2\pi} f^2(z,\varepsilon)dx = \varepsilon^2, \qquad (6)$$

In order to determine this solution we employ a combination of power-series and Fourier-series expansions. Let :

$$f(z,\varepsilon) = \sum_{n=1}^{\infty} f_n(z)\varepsilon^n \qquad (7a)$$

$$\lambda = \sum_{n=0}^{\infty} \lambda_n \varepsilon^n, \lambda_0 = k^2 \qquad (7b)$$

$$\sigma = \sum_{n=1}^{\infty} \sigma_n \varepsilon^n, \qquad (7c)$$

and, since the functions $f_n(z)$ must be periodic of period 2π,

$$f_n(z) = \sum_{n=1}^{\infty} \hat{f}_n(p) \cos pz, \qquad (8)$$

Utilizing (7a,b,c) and (8) in (5), (6), collecting powers of ε and harmonic like terms we obtain:

$$\sigma_n = \frac{1}{4}\sum_{j=1}^{n-1}\sum_{l=1}^{\infty} \hat{f}_j(l)\,\hat{f}_{n-j}(l) \qquad (9a)$$

$$p^2(p^2-1)\hat{f}_n(p) = p^2\lambda_j \hat{f}_1(p) + g(n,p) \qquad (9b)$$

$$\sum_{j=1}^{n-1}\sum_{l=1}^{\infty} \hat{f}_j(l)\,\hat{f}_{n-j+1}(l) = \delta_{n\,2}\,. \qquad (9c)$$

(δ Kronecker symbol)

where $g(n,p)$ in (9b) is a function which depends on $\hat{f}_j(l)$ $1 \le j \le n-1$, $1 \le l \le j$.

Equation (9b) defines λ_{n-1}. Using (9c) with $n = 2$ one obtains $\hat{f}_1(1) = 1$ and therefore, from (9b) when $p = 1$:

$$\lambda_{n-1} = -g(n,1) \qquad (9d)$$

Thus, starting with $n = 2$ and, recursively, one determines λ_{n-1} from (9d), uses (9b) to evaluate $f_n(p)$ for $1 < p \le n$, (9c) then fixes the value of $\hat{f}_n(1)$ and, finally, uses (9a) to calculate σ_n.

For the sake of brevity we refer the reader to reference [7] where all the steps of the algorithm are given in detail.

We have implemented the just described algorithm in a computer program, evaluating all the coefficients for $n \le 50$. Using the ratios $\lambda_{2n}/\lambda_{2(n-1)}$ and $\sigma_{2n}/\sigma_{2(n-1)}$ and extrapolating for $n = 1000$, using rational functions, we determined numerical radii of convergence $\rho_\lambda(k)$ and $\rho_\sigma(k)$. Both agree to more than six decimal places , and therefore we shall indicate this common value by $\rho(k)$. Since the main goal is to solve equation (4) rather than (5), we then determined the value $\varepsilon_1 = \varepsilon_1(k)$ for which $\lambda = 1$, i.e., $\lambda(\varepsilon_1(k)) = 1$. The numerical results indicate that $\varepsilon_1(k) < \rho(k)$ only if $0.63 \le k \le 1$. (Figure 1).

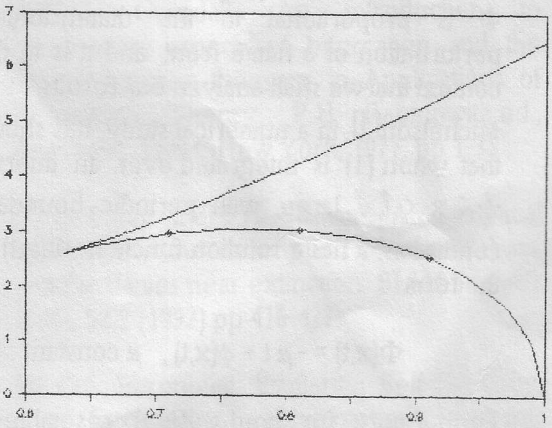

Figure 1. The upper curve represents the common radius of convergence of $\sigma(\varepsilon)$ and $\lambda(\varepsilon)$ obtained by extrapolation to 1000 terms in (7b,c) for $0.63 \le k \le 1$. The lower curve represents $\varepsilon_1(k)$ such that $\lambda(\varepsilon_1(k)) = 1$.

If, rather than using an artificial λ, we consider k as a parameter in (4), it is possible to prove as before that a periodic solution bifurcates from f_0 at $k = k_0 = 1$, $\sigma = 0$. Hence, we proceed as in the previous section, that is, we use (6), (7a),(7b) and (8) in (4), expanding now k^2 in terms of ε as

$$k^2 = \sum_{n=0}^{\infty} k_n \varepsilon^n, \quad k_0 = 1 \tag{10}$$

We obtain in this way a system of equations which can be solved recursively as before. In this case k_{n-1} plays the role of λ_{n-1}.

The numerical results obtained with these expansions are essentially the same as those obtained with the first approach. The present scheme has two advantages over the previous one. First, no artificial parameter is introduced in the equation and, second, the dependence of σ on k is obtained parametrically with just one run of the program. The numerical radii of convergence of (7b) and (10) agree, with common value $\rho = 3.072627$ which determines the minimum value $k = 0.796958$

As an alternative approach to those described before, one can substitute the periodic solutions of (4) by finite Fourier sums

$$f_N(z) = \sum_{p=1}^{N} \hat{f}(p) \cos pz, \tag{11}$$

Collecting like terms, one obtains the system of non-linear algebraic equations :

$$p^2(p^2-1)\hat{f}(p) + \frac{1}{2} \sum_{l=1}^{N-p} l(p+l)\, \hat{f}(l)\hat{f}(p+l) = 0 \tag{12a}$$

$$g_N(p) = \sum_{l=p-N}^{N} l(p-l)\, \hat{f}(l)\hat{f}(p-l) = 0 \tag{12b}$$

which, once solved, defines σ by

$$\sigma = \sigma_N = \frac{1}{4} \sum_{p=1}^{N} p^2 \hat{f}^2(p)$$

Clearly, one can not expect (12a,b), defined for $1 \le p \le N$ and $N < p \le 2N$ respectively, to have non-trivial solutions because it is a system of 2N equations with N unknowns. However, if the square system (12a) has a non-trivial solution, then, the L^2 norm of the residual R_N is

$$R_N = \{ \sum_{p=N}^{2N} g_N^2(p) \}^{1/2}$$

If R_N is small, then one can use (11) as a good approximation to the solution of (4). This algorithm, when implemented, reproduced all the numerical results obtained with the previous approaches, within their range of validity, with very small residuals . This scheme is faster than the others, mainly because one small number of terms, N = 10 for example, gives results which are not different from those obtained with N = 100. The validity of the approximation is no longer good as k approaches 0.5. Using this method we determined the parametric curve $(\sigma(k), \varepsilon(k))$ where ε is the L^2 norm of the approximate solution (i.e., directly related to the amplitude of the periodic solution) shown inf Figure 2. One can observe in it that the maximum of $\varepsilon(k)$ is attained at $k = 0.78$ with $\varepsilon = 3.068126$ and $\sigma = 1.498951$.The linear analysis of the stability of the zero solution of (1) yields $k = \sqrt{2}/2$ as the wave-number which gives the maximal amplification. The maximum flame velocity, $\sigma = 1.6030$, is attained at a larger value of k, namely, $k = 0.84$ where $\varepsilon = 2.972199$, .It is interesting to observe that two flames can travel with the same velocity, although presenting different periods and amplitudes.

2. PROGRESSIVE WAVES

Let ω be an arbitrary real number, $f(z)$ a 2π-periodic solution of (4) for a given k and $\sigma = \sigma(k)$ the corresponding value of σ. Let $\phi(x,t)$ be the function defined by

$$\phi(x,t) = f(k(x - \omega t)).$$

Straightforward substitution shows that

$$\Phi(x,t,\omega) = \omega x - \left(\frac{\omega^2}{2} + \sigma\right) t + \phi(x,t) \quad (13)$$

is a solution of (1)

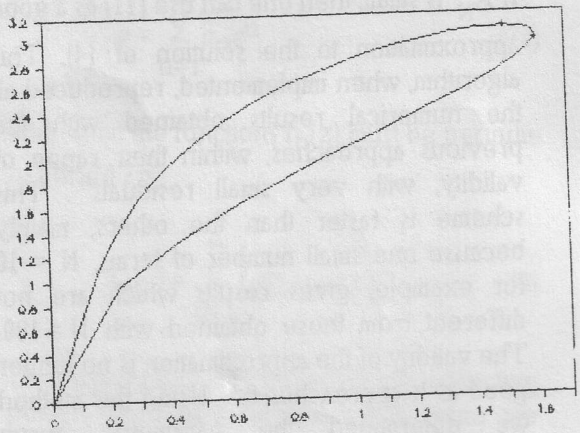

Figure 2. L^2 norm and flame velocity of cellular flames as a functions of the wave-number k.

Since $\Phi(x,t,\omega)$ is a one-parameter family of solutions of (1), its envelope, defined by the equation

$$\frac{\partial \Phi}{\partial \omega} = x - \omega t - k\, t\, f'(k(x - \omega t)) = 0$$

is also a solution of the differential equation.

Using f and k obtained with the first perturbation scheme, expanding ω in power series in ε, with

$$\omega_0 = \omega(0,x,t) = x/t,$$

it becomes evident that all higher order terms in ε do not depend on x, hence, we can write

$$\omega = x/t + W(t)$$

which, in turn, yields the envelope

$$\Phi(x,t) = \Phi_E(x,t) = \frac{x^2}{2t} - \ln t \quad (14)$$

Equation (1) could also be solved using a self-similarity technique.Following Barenblatt[8], if one sets $x = \ln z$, $t = \ln \tau$, the resulting equation possesses the solution, after returning to the original variables,

$$\Phi(x,t) = \Phi_{ES}(x,t) = \frac{x^2}{2t},$$

which is the envelope of the one parameter, plane-flames solutions

$$\Phi(x,t) = \Phi_S(x,t) = \omega x - \frac{\omega^2}{2} t,$$

The logarithmic term in (14) is then due to the progressive wave. Although (14) is a solution of (1), it can not be regarded as a flame, but rather as a boundary of all possible progressive waves obtained by space and time periodic perturbations. We must emphasize that this boundary travels in time at a slower velocity than the average velocity of each individual progressive wave.

It can be shown that the progressive wave $\Phi(x,t)$ travels at a slower pace than the travelling wave $\phi(x,t)$ if $\omega > \sqrt{\sigma}$. If, on the other hand, $\omega < \sqrt{\sigma}$ it goes faster. The velocity is exactly ω if $\omega = \sqrt{\sigma}$. In this case, we can write

$$\Phi(x,t) = \sqrt{2\sigma}(x - \sqrt{2\sigma}t) + f(k(x - \sqrt{2\sigma}t)).$$

Let h be a fixed,but otherwise arbitrary real number, and consider the solutions $x = x(t)$ of the equation

$$\Phi(x,t) = \omega x - \left(\frac{\omega^2}{2}+\sigma\right)t + f(k(x-\omega t)) = h \qquad (15)$$

The total derivative of (15) with respect to t yields

$$\dot{x}(t) = \frac{dx(t)}{dt} = \frac{\left(\frac{\omega^2}{2}+\sigma\right) + k\,\omega f'}{\omega + k f'}.$$

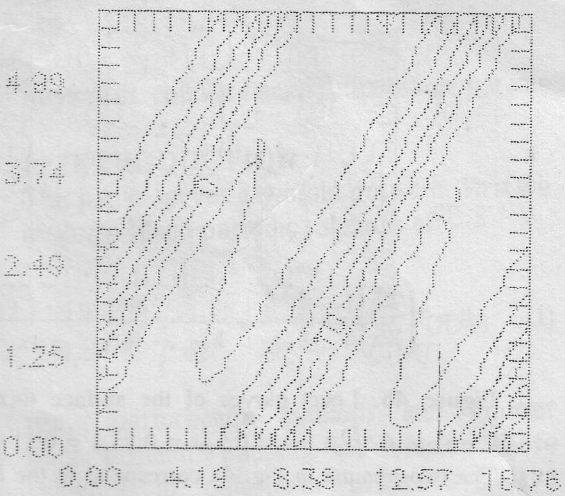

Figure 3a. Level curves of the surface $\Phi(x,t)$. Case $\omega < \sqrt{2\sigma}$. k=0.75, ω = 1.4, σ = 1.381398

Therefore, if $-\omega/k < \min f'(z)$, $z \in [0, 2\pi]$ and $\omega < \sqrt{2\sigma}$ then, the curve x(t) is monotonic, otherwise it has infinite or zero slope at a finite time, depending on whether one or both inequalities are violated (Figures 3a and 3b).

This can be better understood with the help of Figure 3a in which $\omega < \sqrt{2\sigma}$. The level curves shown in that figure decrease from right to left, then, if one fixes the level at, say, h = 0, then one would only see fresh gas. At t $\approx$ 0.25 one point of light, A, would appear at z = 0, the flame front. This point would travel toward the right until t $\approx$1, when another flame front would appear at z $\approx$ 4.19. This second point will split in two fronts, B and C, travelling C to the right and B to the left for a while, turning B toward the right some time after its appearance. After this instant one would observe that in the leftmost region of fresh gas, the point B would be advancing into the burned gas. Physically this, in an actual experiment, would be interpreted as a moving mass of gas going to the right and the observed velocities of A and B would be relative velocities. At t $\approx$ 4 the point A would reach the point B and both would deasapear, that is, the gas bubble collapses while the point C keeps moving toward the right. In Figure 3b $\omega > \sqrt{2\sigma}$. Although the description

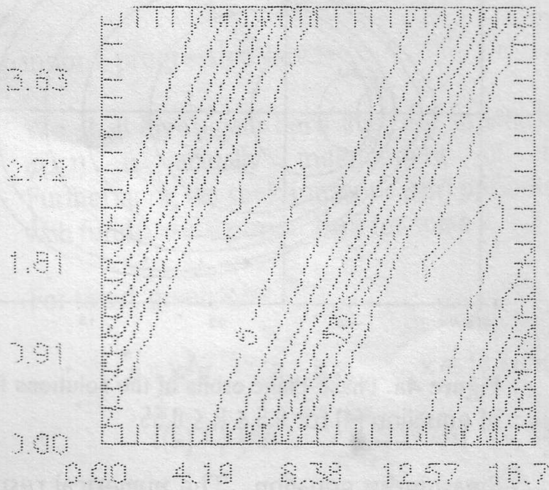

Figure 3b. Level curves of the surface $\Phi(x,t)$. Case $\omega > \sqrt{2\sigma}$. k = 0.75, ω = 1.925, σ = 1.381398

of the phenomenon would be similar if we choose h = 5, in this case, the point B will reach the point C and the bubble of fresh gas would join the region of fresh gas to the right of C.

A very interesting phenomenon takes place when 0.5 < k < 0.55. In this region f' has two local minima in $0 \le z \le 2\pi$, Figure 4a . Figure 4b shows what happens in this cases, when $-\omega/k < 0$ is greater than the greatest of both minima of f', both regions where f' is negative generate bent level curves, while the narrow segment where f' > 0 seems to yield better behaved curves.

4. DISCUSSION OF RESULTS AND CONCLUSIONS

We have presented three schemes for solving the Kuramoto-Sivashinsky equation, determined space-periodic, constant-velocity solutions and progressive waves solutions. The three schemes, within their limitations, proved to be fast and accurate. The fact that they solve the Kuramoto-Sivashinsky equation is encouraging for their application toward the solution of the full Michelson-

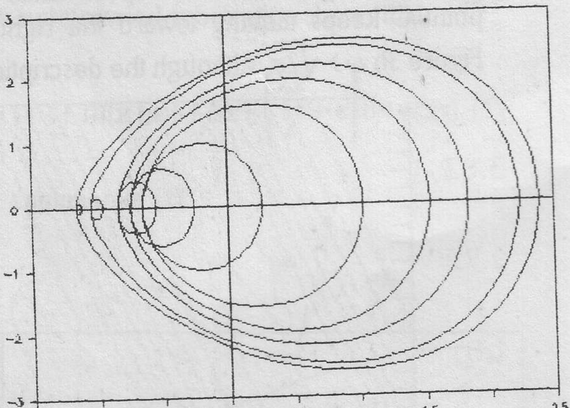

Figure 4a. Phase-Plane orbits of the solutions f(z) of equation (4) for $0.5 \leq k \leq 0.55$

Sivashinsky equation . The numerical results here obtained agree with those previously found by other authors , for example [4].

The progressive waves found in this paper seem to be new in the literature. Their behavior describe fairly well the expected physical behavior and, once again, this opens new possibilities for similar equations, like those modelling thin films [10].

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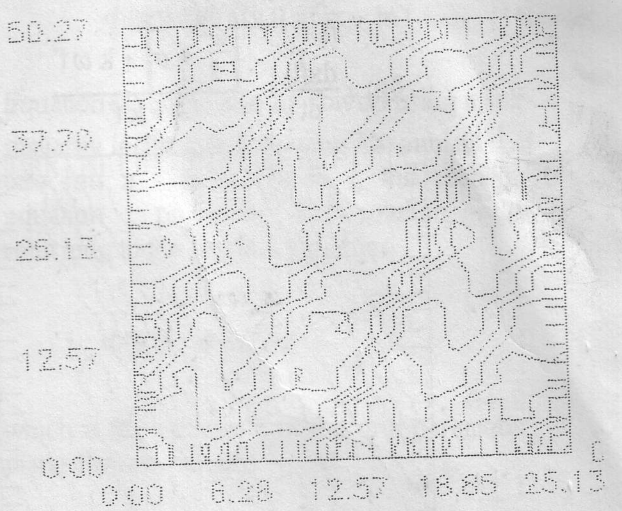

Figure 4b. Level curves of the surface $\Phi(x,t)$. Case $\omega < \sqrt{2\sigma}$, $k = 0.5, \omega = 0.25, \sigma = 0.11147$. The two minima in Fig. 4a correspond to the two perturbed zones.

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