SOME SECOND ORDER EFFECTS IN GYROSCOPES INFLUENCED BY DRAG

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1. Introduction

Consider a gyroscope consisting of a rotor and two gimbals (see Fig. 1).

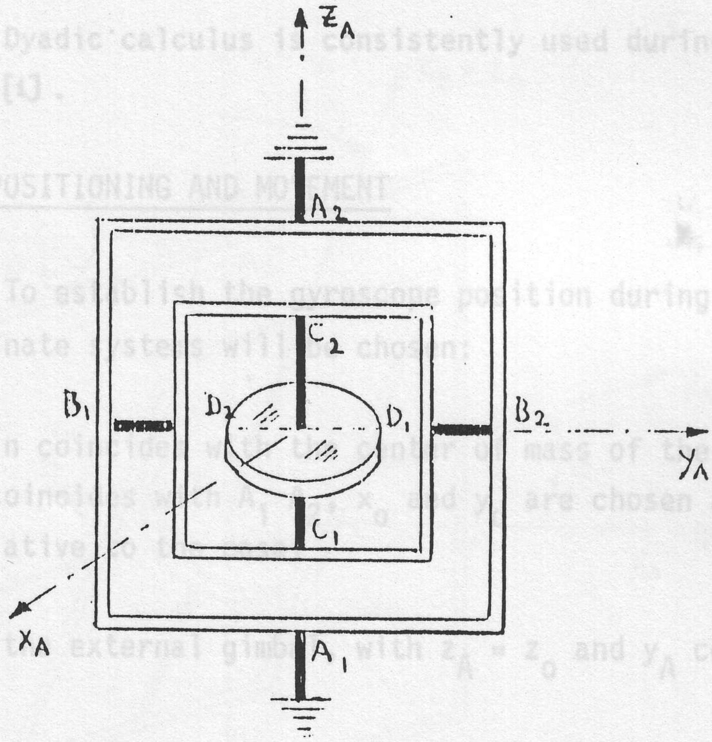


Fig. 1 - Reference position of a gyroscope.

The external gimbal is connected with an external case at points A_1 and A_2 by a pair of axes. At points B_1 and B_2 the external gimbal is connected with the internal one by a second pair of axes.

Finally, the rotor is connected at points C_1 and C_2 with the internal gimbal by a third pair of axes.

All axes allow free rotation.

The aim of this work is to establish the formulation of equations expressing the influence of

(i) gimbal motion (gimbals not considered massless), and
(ii) air drag

on the rotor motion.

Dyadic calculus is consistently used during the derivation of equations, [1].

2) GYROSCOPE POSITIONING AND MOVEMENT

To establish the gyroscope position during its movement, several coordinate systems will be chosen:

S_o - The origin coincides with the center of mass of the gyroscope. z_o axis coincides with A_1 A_2, x_o and y_o are chosen arbitrarily, fixed relative to the case.

S_A - Fixed on the external gimbal, with $z_A = z_o$ and y_A coinciding with B_1 B_2.

S_B - Fixed on the internal gimbal, with $y_B = y_A$ and z_B coinciding with C_1 C_2.

S_R - Fixed on the rotor: $z_R = z_B$; x_R and y_R in the rotor plane.

The gyroscope can be conducted from its position of reference, S_o, to its ambitrary position, S_R by means of:

(i) Rotation (of the external gimbal) through angle ϕ, around $A_1 A_2$.

(ii) Rotation (of the internal gimbal) through angle Θ, around $B_1 B_2$

(iii) Rotation (of the rotor) through ψ around $C_1 C_2$.

The angles ϕ, Θ, Ψ (known as Euler Angles) determine completely the gyroscope position at any instant. Their rates $\dot{\phi}$, $\dot{\Theta}$, $\dot{\Psi}$ determine, respectively, angular velocites of

(i) precession
(ii) nutation
(iii) spin

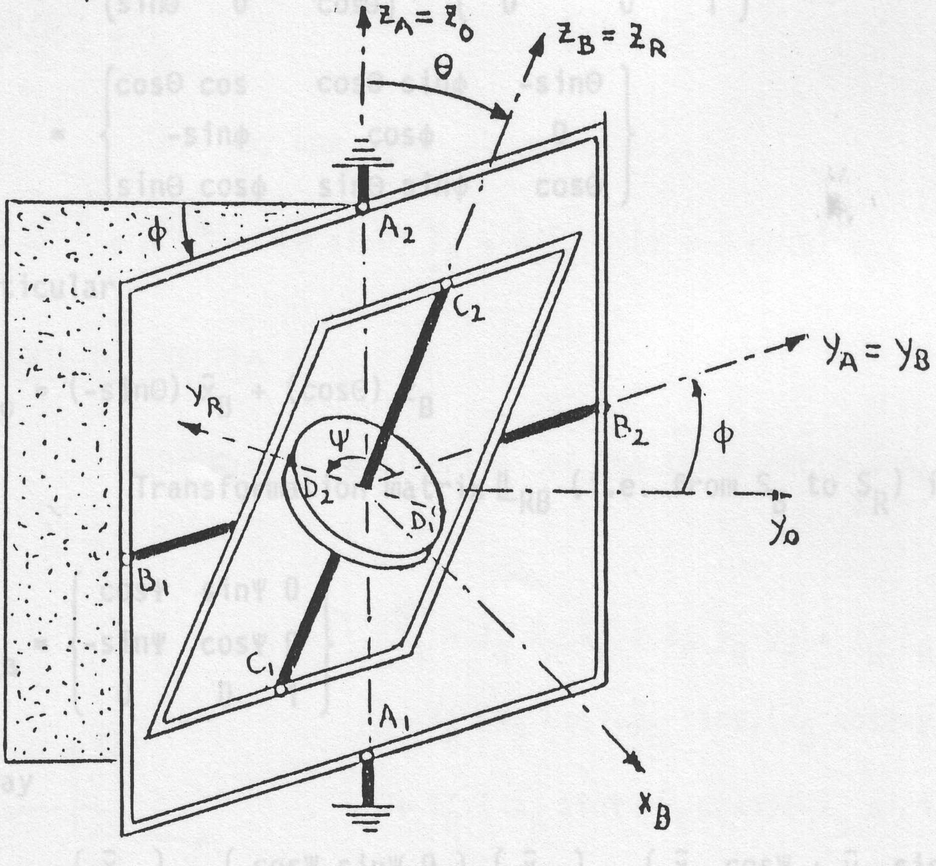


Fig. 2 - ϕ, Θ and Ψ.

3) COORDINATE SYSTEM TRANSFORMATIONS

Observe that transformation $S_o \rightarrow S_B$ consists of

(i) rotation through ϕ around $\hat{z}_o$

(ii) rotation through Θ around the new position of $\hat{y} = \hat{y}_A$

This way, the transformation matrix $\mathbb{L}_{BA}$ (i.e. from S_A to S_B) is

$$\mathbb{L}_{BA} = \begin{Bmatrix} \cos\Theta & 0 & -\sin\Theta \\ 0 & 1 & 0 \\ \sin\Theta & 0 & \cos\Theta \end{Bmatrix} \begin{Bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{Bmatrix} =$$

$$= \begin{Bmatrix} \cos\Theta \cos & \cos\Theta \sin\phi & -\sin\Theta \\ -\sin\phi & \cos\phi & 0 \\ \sin\Theta \cos\phi & \sin\Theta \sin\phi & \cos\Theta \end{Bmatrix} \tag{1}$$

In particular

$$\hat{z}_o = (-\sin\Theta)\, \hat{x}_B + (\cos\Theta)\, \hat{z}_B \tag{2}$$

Transformation matrix $\mathbb{L}_{RB}$ (i.e. from S_B to S_R) is

$$\mathbb{L}_{RB} = \begin{Bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

This way

$$\begin{Bmatrix} \hat{x}_R \\ \hat{y}_R \\ \hat{z}_R \end{Bmatrix} = \begin{Bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{Bmatrix} \begin{Bmatrix} \hat{x}_B \\ \hat{y}_B \\ \hat{z}_B \end{Bmatrix} = \begin{Bmatrix} \hat{x}_B \cos\Psi + \hat{y}_B \sin\Psi \\ -\hat{x}_B \sin\Psi + \hat{y}_B \cos\Psi \\ \hat{z}_B \end{Bmatrix}$$

4) ANGULAR VELOCITIES

The angular velocity of S_B relative to the case (i.e. S_A) is

$$\vec{\omega}^{B/0} = \dot{\phi}\,\hat{z}_A + \dot{\Theta}\,\hat{y}_B \qquad (3)$$

In particular, substituting (2) into (3), one gets

$$\vec{\omega}^{B/0} = -(\sin\Theta)\,\dot{\phi}\,\hat{x}_B + \dot{\Theta}\,\hat{y}_B + (\cos\Theta)\,\dot{\phi}\,\hat{z}_B \qquad (4)$$

The angular velocity of the rotor relative to S_B' is

$$\vec{\omega}^{R/B} = \dot{\Psi}\,\hat{z}_B$$

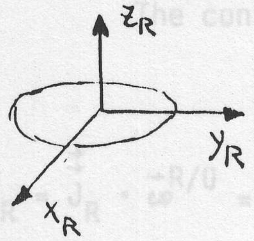
This way one gets

$$\vec{\omega}^{R/0} = \vec{\omega}^{R/B} + \vec{\omega}^{B/A} = -(\dot{\phi}\sin\Theta)\,\hat{x}_B + \dot{\Theta}\,\hat{y}_B + (\dot{\Psi} + \dot{\phi}\cos\Theta)\,\hat{z}_B \qquad (5)$$

5) MOMENTS OF INERTIA

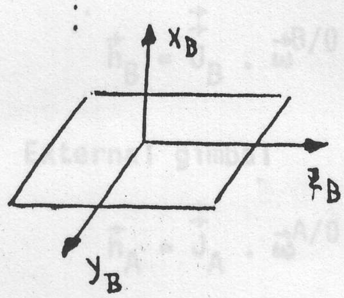
The dyadic of inertia of the three parts are:

Rotor



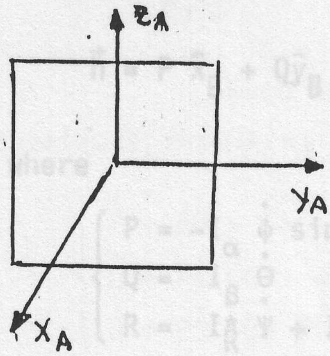
$$\begin{aligned}
\vec{\vec{J}}_R &= I'_R\,\hat{x}_R\hat{x}_R + I'_R\,\hat{y}_R\hat{y}_R + I^*_R\,\hat{z}_R\,\hat{z}_R \\
&= I'_R\,(\hat{x}_B\cos\Psi + \hat{y}_B\sin\Psi)(\hat{x}_B\cos\Psi + \hat{y}_B\sin\Psi) + \\
&\quad + I'_R\,(-\hat{x}_B\sin\Psi + \hat{y}_B\cos\Psi)(-\hat{x}_b\sin\Psi + \hat{y}_B\cos\Psi) + \\
&\quad + I^*_R\,\hat{z}_B\,\hat{z}_B \\
&= I'_R\,\hat{x}_B\,\hat{x}_B + I'_R\,\hat{y}_B\,\hat{y}_B + I^*_R\,\hat{z}_B\,\hat{z}_B
\end{aligned} \qquad (6)$$

Internal gimbal



$$\vec{\vec{J}}_B = I_B^* \hat{x}_B \hat{x}_B + I_B' \hat{y}_B \hat{y}_B + I_B'' \hat{z}_B \hat{z}_B \quad (7)$$

External gimbal



$$\vec{\vec{J}}_A = I_A^* \hat{x}_A \hat{x}_A + I_A' \hat{y}_A \hat{y}_A + I_A'' \hat{z}_A \hat{z}_A \quad (8)$$

6) ANGULAR MOMENTUM

One has

$$\vec{h} = \int \vec{R} \times \dot{\vec{R}}\, dm = \sum_i \left(\int_{m_i} \vec{R} \times \dot{\vec{R}}\, dm \right) = \sum \vec{h}_i$$

This means that the contributions of all parts are additive

The contribution of

Rotor

$$\vec{h}_R = \vec{\vec{J}}_R \cdot \vec{\omega}^{R/0} = (I_R' \hat{x}_B \hat{x}_B + I_R' \hat{x}_B \hat{x}_B + I_R^* \hat{z}_B \hat{z}_B) \cdot$$

$$\cdot \left[-\hat{x}_B \dot{\phi} \sin\Theta + \hat{y}_B \dot{\Theta} + \hat{z}_B (\dot{\Psi} + \dot{\phi} \cos\Theta) \right]$$

$$= \hat{x}_B (-I_R' \dot{\phi} \sin\Theta) + \hat{y}_B (I_R' \dot{\Theta}) + \hat{z}_B [I_R^* (\Psi + \dot{\phi} \cos\Theta)] \quad (9)$$

Internal gimbal

$$\vec{h}_B = \vec{\vec{J}}_B \cdot \vec{\omega}^{B/0} = -\hat{x}_B \, I_B^* \, \dot{\phi} \sin\Theta + \hat{y}_B \, I_B' \, \dot{\Theta} + z_B \, I_B'' \, \dot{\phi} \cos\Theta \qquad (10)$$

External gimbal

$$\vec{h}_A = \vec{\vec{J}}_A \cdot \vec{\omega}^{A/0} = (I_A^* \, \hat{x}_A \, \hat{x}_A + I_A' \, \hat{y}_A \, \hat{y}_A + I_A'' \, \hat{z}_A \, \hat{z}_A) \cdot (\dot{\phi} \, \hat{z}_A)$$

$$= I_A'' \, \dot{\phi} \, z_A = - \hat{x}_B \, I_A'' \, \dot{\phi} \sin\Theta + \hat{z}_B \, I_A'' \, \dot{\phi} \cos\Theta \qquad (11)$$

This way, the total angular momentum can be written

$$\vec{h} = P \, \hat{x}_B + Q\hat{y}_B + R\hat{z}_B \qquad (12)$$

where

$$\begin{cases} P = -I_\alpha \, \dot{\phi} \sin\Theta \\ Q = I_\beta \, \dot{\Theta} \\ R = I_R^* \dot{\Psi} + I_\gamma \, \dot{\phi} \cos\Theta \end{cases}$$

and where

$$\begin{cases} I_\alpha = I_R' + I_B^* + I_A'' \\ I_\beta = I_R' + I_B' \\ I_\gamma = I_R^* + I_B'' + I_A'' \end{cases}$$

7) <u>EQUATIONS OF MOVEMENT (EULER EQUATIONS)</u>

As is well known, one has

$$\dot{\vec{h}} = \vec{G}$$

where

$\vec{h}$ = total angular momentum

$\vec{G}$ = resultant of external torques and where the symbol ($\dot{\vec{\ }}$) denotes the inertial time deviration (d/dt).

Expressing equations in a non-inertial system, say S_B, one has:

$$\dot{\vec{h}} = \dot{\vec{h}}^B + w^{B/0} \times \vec{h} \tag{14}$$

where the symbol $(\dot{\vec{\ }})^B$ denotes the time derivative as determined by the observer in S_B, that is - see eq. (12):

$$\dot{\vec{h}}^B = \dot{p}\,\bar{x}_B + \dot{Q}\,\bar{y}_B + \dot{R}\,\bar{z}_B$$

Even though this is a standard procedure of getting Euler equations, resulting in:

$$\begin{cases} \dot{P} + R\dot{\Theta} - Q\dot{\phi}\cos\Theta = G_{x_B} \\ \dot{Q} + P\dot{\phi}\cos\Theta + R\dot{\phi}\sin\Theta = G_{y_B} \\ \dot{R} + P\dot{\Theta} - Q\dot{\phi}\sin\Theta = G_{z_B} \end{cases} \tag{15}$$

or, after some algebra, in

$$\begin{cases} -I_\alpha \ddot{\phi}\sin\Theta + (I_\gamma - I_\alpha - I_\beta)\dot{\phi}\dot{\Theta}\cos\Theta + I_R^*\dot{\Psi}\dot{\Theta} = G_{x_B} \\ I_\beta \ddot{\Theta} + (I_\gamma - I_\alpha)\dot{\phi}^2\cos\Theta\sin\Theta + I_R^*\dot{\Psi}\dot{\phi}\sin\Theta = G_{y_B} \\ I_R \ddot{\Psi} + I_\gamma \ddot{\phi}\cos\Theta + (I_\alpha - I_\beta - I_\gamma)\dot{\phi}\dot{\Theta}\sin\Theta = G_{z_B} \end{cases}$$

one gets, this way, second order equations which are not immediately integrable.

A more convenient way is to express $\vec{h}$ in the inertial system, S_o, then no $\vec{\omega} \times \vec{h}$ term will be required and equations of motion will be directly integrable.

Of course, one has

$$(\vec{h})_o = \mathbb{L}_{OB}\,(\vec{h})_B$$

where $(\vec{h})_0$ and $(\vec{h})_B$ express column matrices composed of the components of angular momentum ; S_o and S_B, respectively.

Thus, see eq. (1)

$$(\vec{h})_0 = \begin{Bmatrix} \cos\Theta\cos\phi & -\sin\phi \\ \cos\Theta\sin\phi & \cos\phi \\ -\sin\Theta & 0 \end{Bmatrix} \begin{Bmatrix} \sin\Theta\cos\phi \\ \sin\Theta\sin\phi \\ \cos\Theta \end{Bmatrix} \begin{Bmatrix} P \\ Q \\ R \end{Bmatrix}$$

After some algebra, Euler equations can be written as

$$\frac{d}{dt}\left\{(I_\gamma - I_\alpha)\,\dot{\phi}\cos\Theta\sin\Theta\cos\phi - I_\beta\,\dot{\Theta}\sin\phi + I_R^*\,\dot{\Psi}\sin\Theta\cos\phi\right\} = G_{x_0}$$

$$\frac{d}{dt}\left\{(I_\gamma - I_\alpha)\,\dot{\phi}\cos\Theta\sin\Theta\sin\phi + I_\beta\,\dot{\Theta}\cos\phi + I_R^*\,\dot{\Psi}\sin\Theta\sin\phi\right\} = G_{y_o}$$

$$\frac{d}{dt}\left\{\phi\,(I_\alpha\sin^2\Theta + I_\gamma\cos^2\Theta) + I_R^*\,\Psi\cos\Theta\right\} = G_{z_o} \qquad (17)$$

The above equations can be, now, immediately integrated and, since they will result in a system of linear equation in $\dot{\phi}, \dot{\Theta}, \dot{\Psi}$, can also be solved with respect of these variable.

8) TIME RATES OF EULER ANGLES WITH EXTERNAL TORQUES ABSENT

Assuming $G_{x_0} = G_{y_0} = G_{z_0} = 0$, integrating the above equations and solving them for $\dot{\phi}, \dot{\Theta}, \dot{\Psi}$, one gets, after some manipulations.

$$\dot{\Theta} = \frac{1}{I_\beta}(c_1\sin\phi + c_2\cos\phi)$$

$$\dot{\phi} = \frac{1}{I_\alpha\sin\Theta}\{\,c_3\sin\Theta - (c_1\cos\phi + c_2\sin\phi)\cos\Theta\,\}$$

$$\dot{\Psi} = \frac{1}{I_R^*\sin\Theta}\{\,c_1\cos\phi + c_2\sin\dot{\phi})(\sin^2\Theta + \frac{I_\gamma}{I_\alpha}\cos^2\Theta) + c_3\frac{I_\alpha - I_\gamma}{I_\alpha}\sin\Theta\cos\Theta\}$$

(18)

c_1, c_2 and c_3 can be determined from initial conditions. Wthout loss of generality, one can assume $c_1 = 0$.

9) AIR DRAG DUE TO THE ROTATING DISK

Rotating disk produces a moment due to the air drag in $\hat{z}_R$ direction.

To estimate the drag in turbulent flow (which is our case), choose 1/7 power for velocity distribution.

Centrifugal force per unit volume is $\rho r \omega^2$ [2] (ω = angular velocity) and the centrifugal force acting on a volume dr x ds x δ (δ = comdary layer thickness) becomes $\rho r \omega^2$ ds dr δ. The shearing stress τ_0 forms an angle Θ with the tangential direction and its radial component must balance the centrigal force. Hence

$$\tau_0 \sin\Theta \, dr \, ds = \rho r \omega^2 \delta \, dr \, ds$$

or

$$\tau_0 \sin\Theta = \rho r \omega^2 \delta$$

Using analogy with flat plate, one has

$$\tau_0 \cos\Theta \sim \rho(\omega r)^{7/4} (\nu/\delta)^{1/4}$$

(U_∞ substituted by $r\omega$). Then

$$\delta \sim r^{3/5} (\nu/\omega)^{1/5}$$

The torque becomes

$$M \sim \tau_0 R^3 \sim \rho R \omega^2 (\nu/\omega)^{1/5} R^{3/5} R^3$$

or

$$M \sim \rho\, U^2\, R^3 \left(\frac{\nu}{UR}\right)^{1/5}$$

where

$$U = \omega R.$$

Von Karman using the 1/7 power law for the variation of the tangential velocity component through the boundary layer showed that, for a disk wetted on both sides, the viscous torque is equal to

$$2M = 0.073\, \rho \omega^2\, R^5 \left(\frac{\nu}{\omega R^2}\right)^{1/5} \qquad (19)$$

Thus, c_M becomes

$$c_M = \frac{0.146}{(Rey)^{1/5}}$$

This result shows good agreement for $Rey > 3 \times 10^5$.

REFERENCES:

1. GOLDSTEIN, H. - Classical Mechanics, II ed. Addison-wesley, 1980.

2. SCHLICHTING, H. - Boundary-Layer Theory, VII ed. McGraw-Hill, 1979.

FI vs TIME

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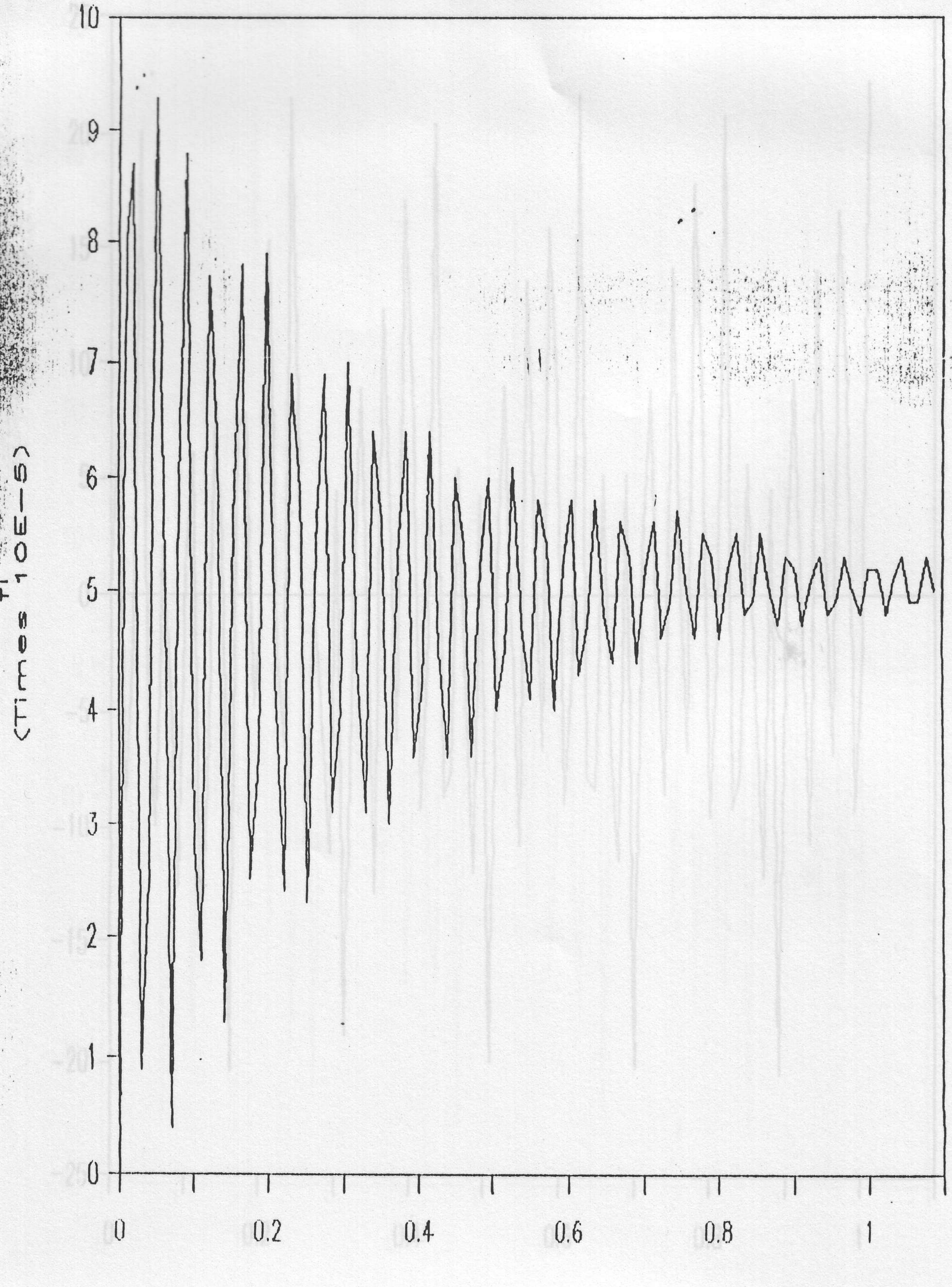
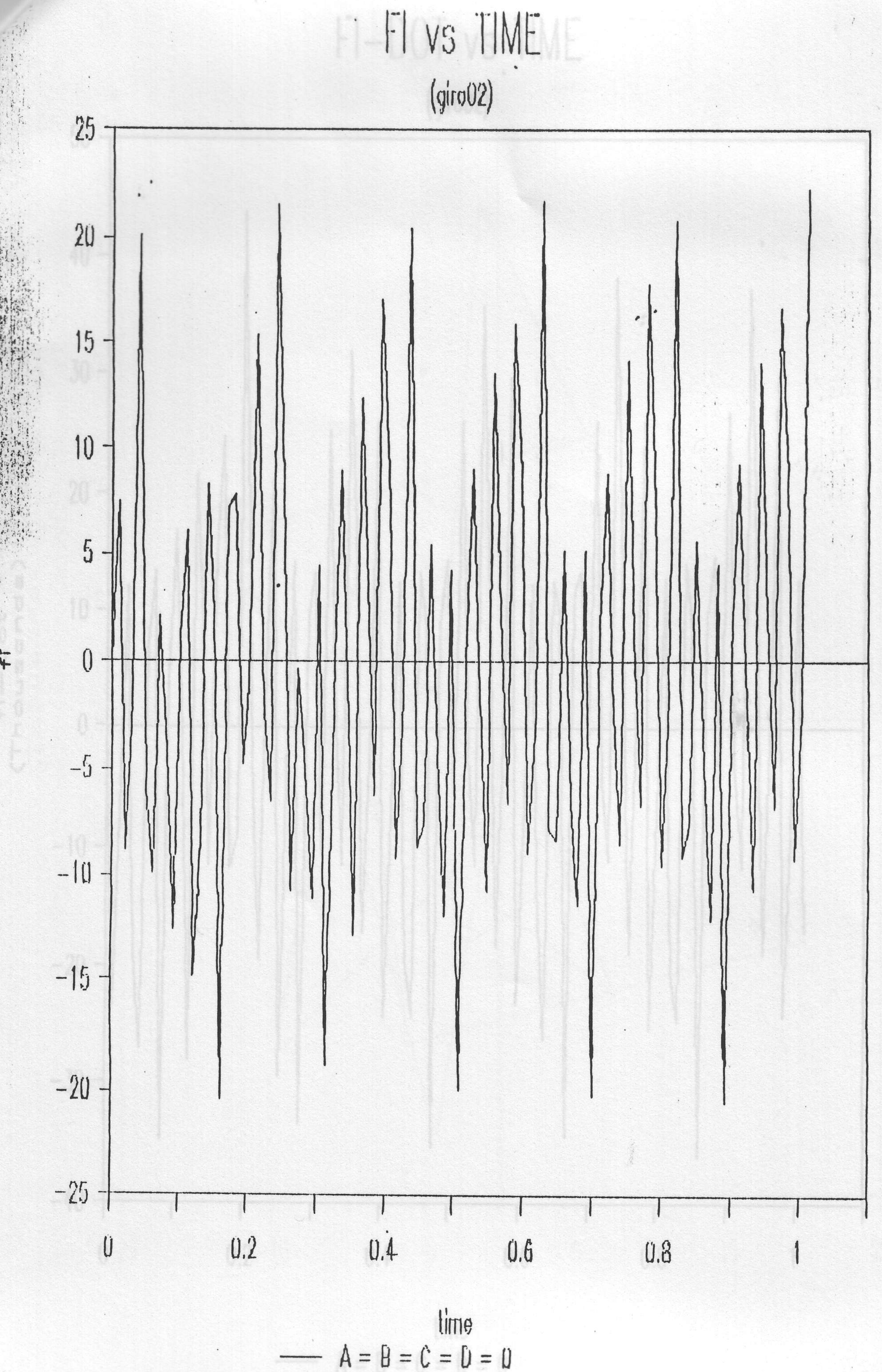
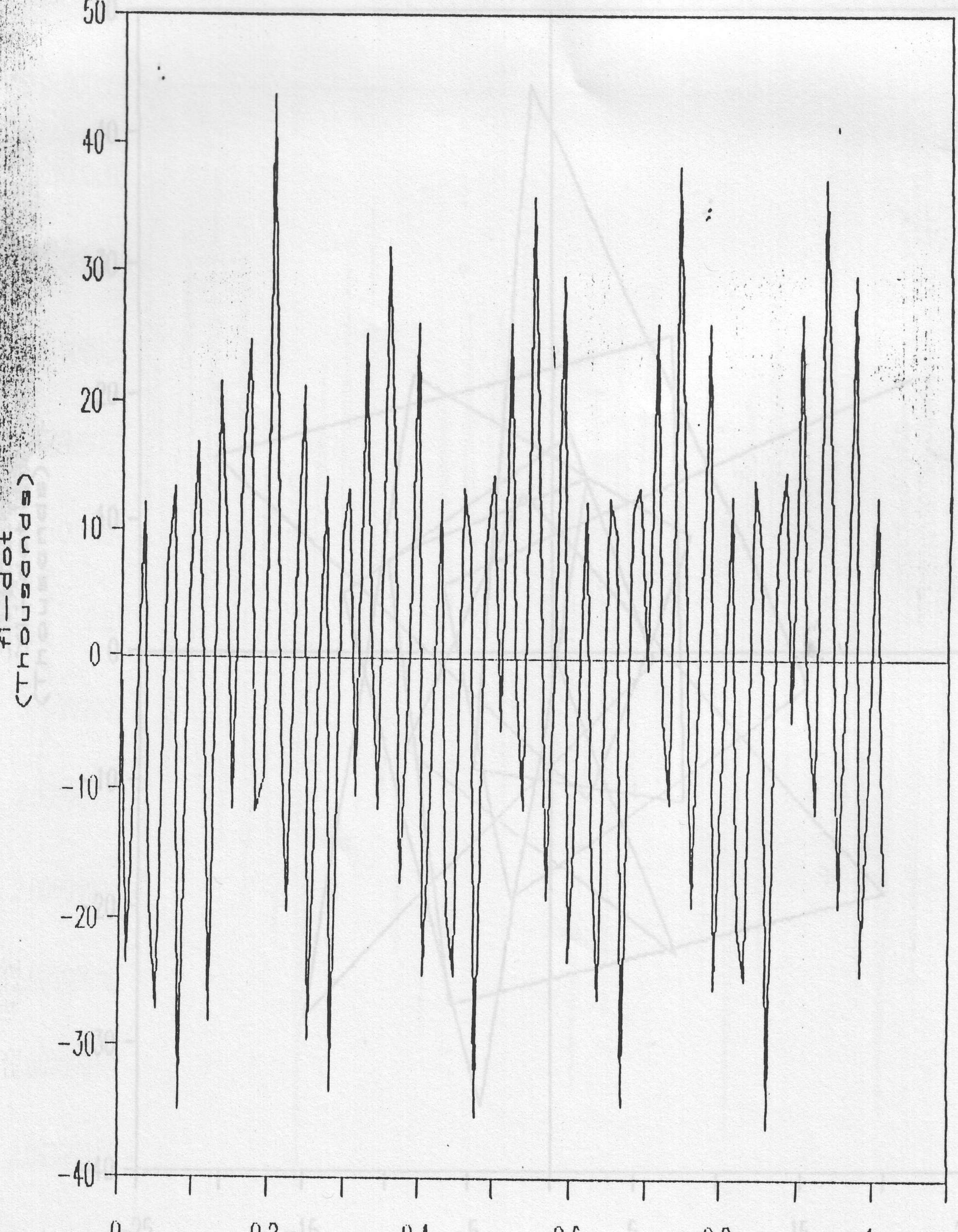
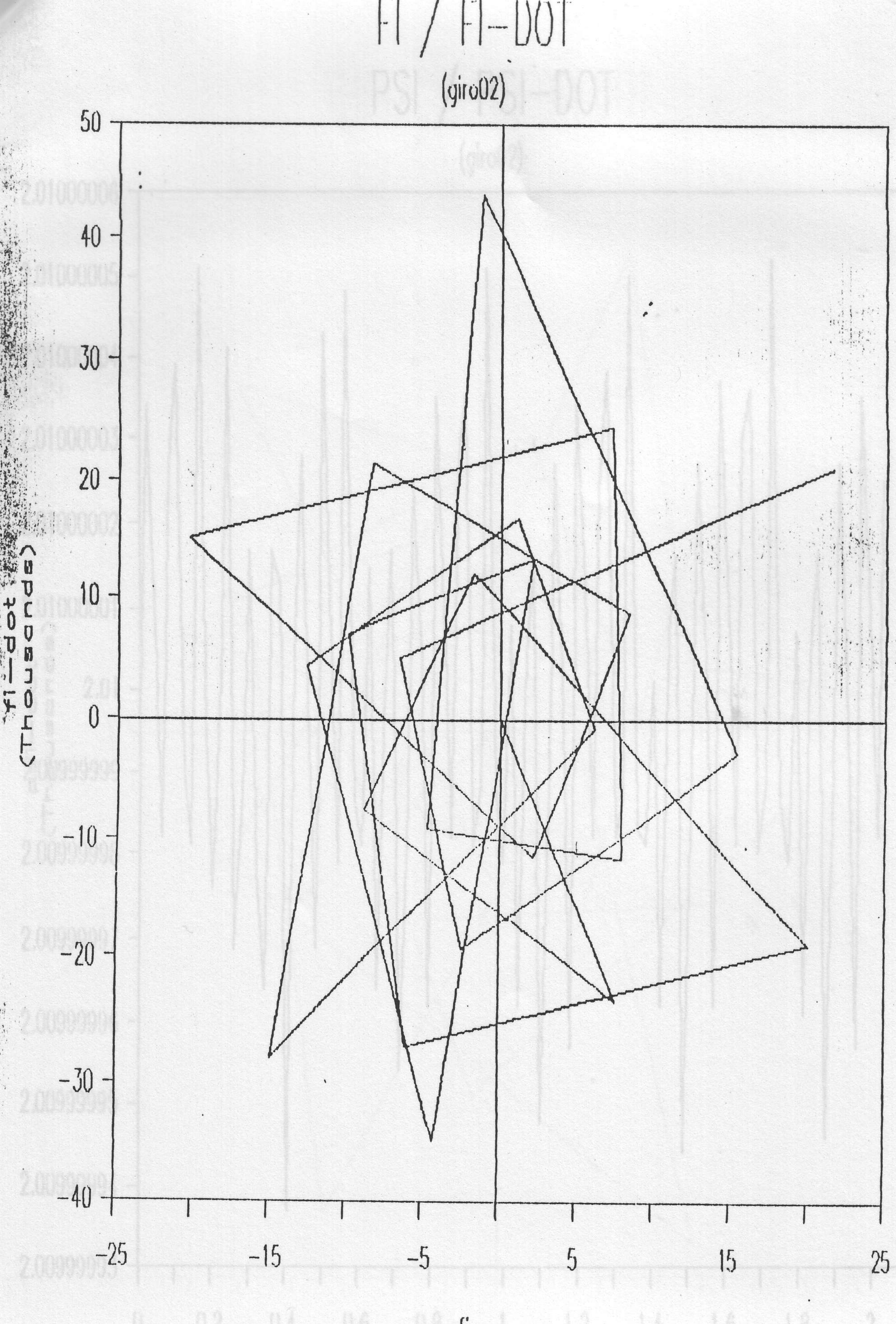


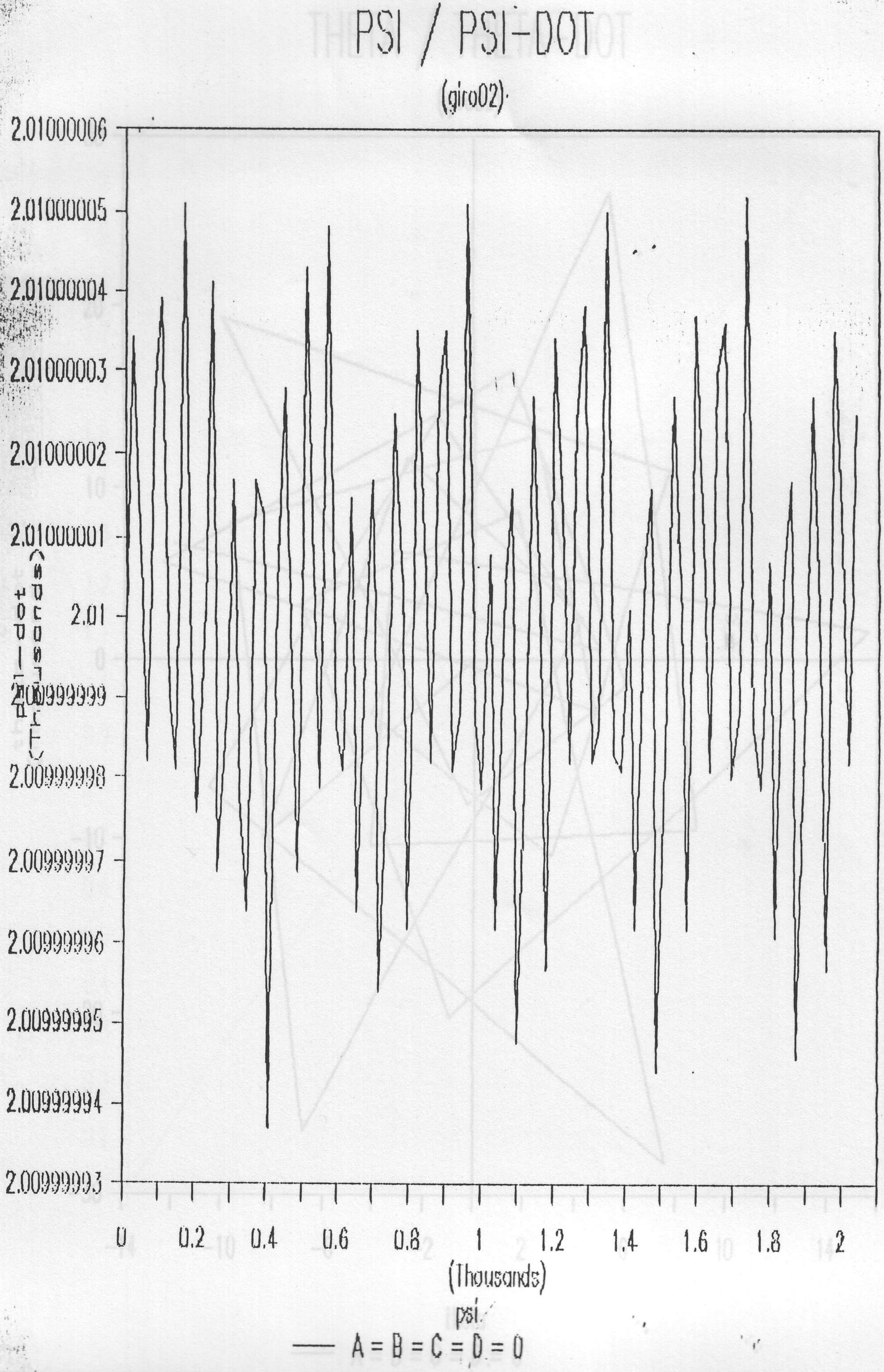

FI vs TIME
(giro02)
25
20
15
10
5
0
-5
-10
-15
-20
-25
FI
0
0.2
0.4
0.6
0.8
1
time
A = B = C = D = 0

FI-DOT vs TIME

(giro02)

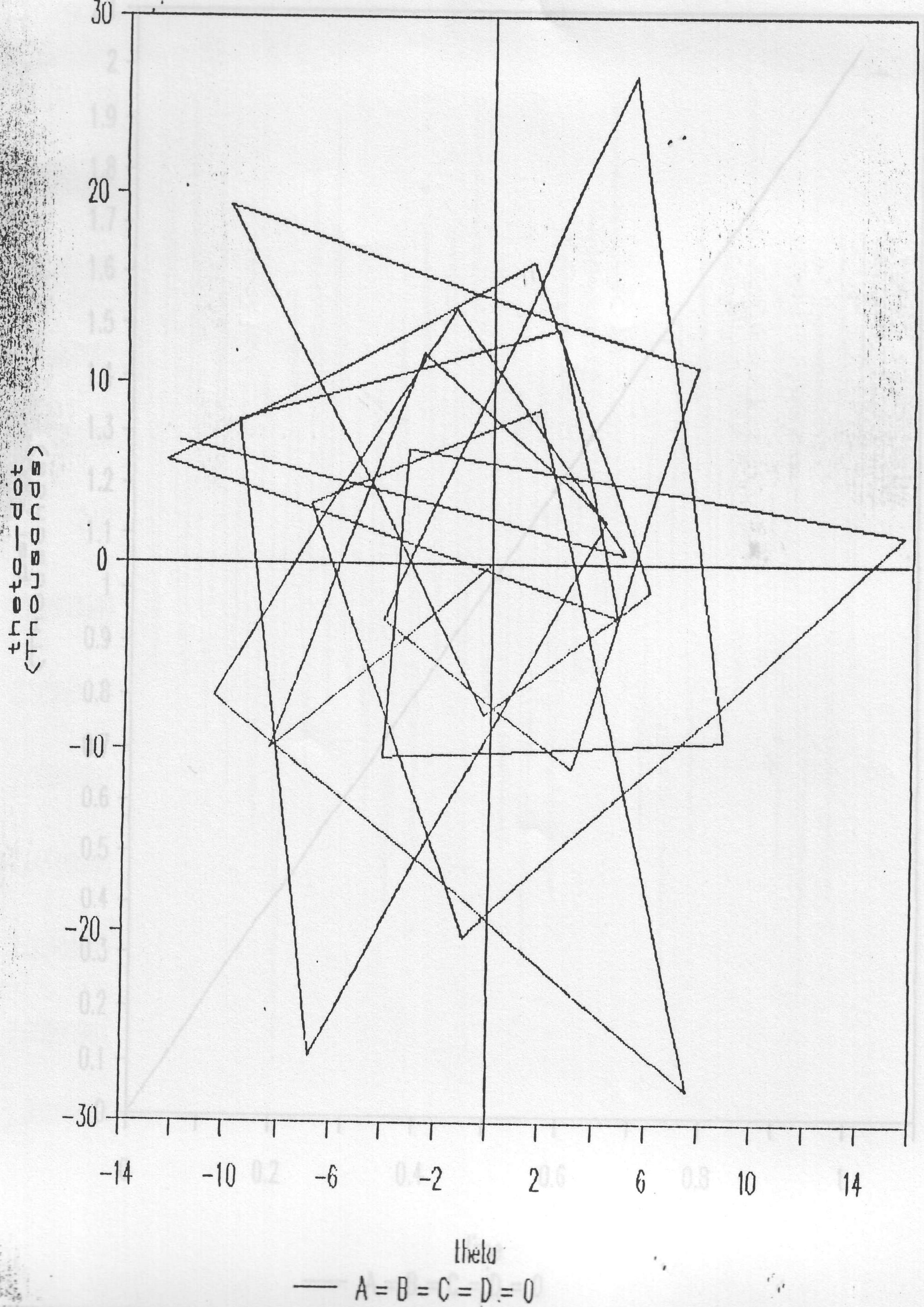


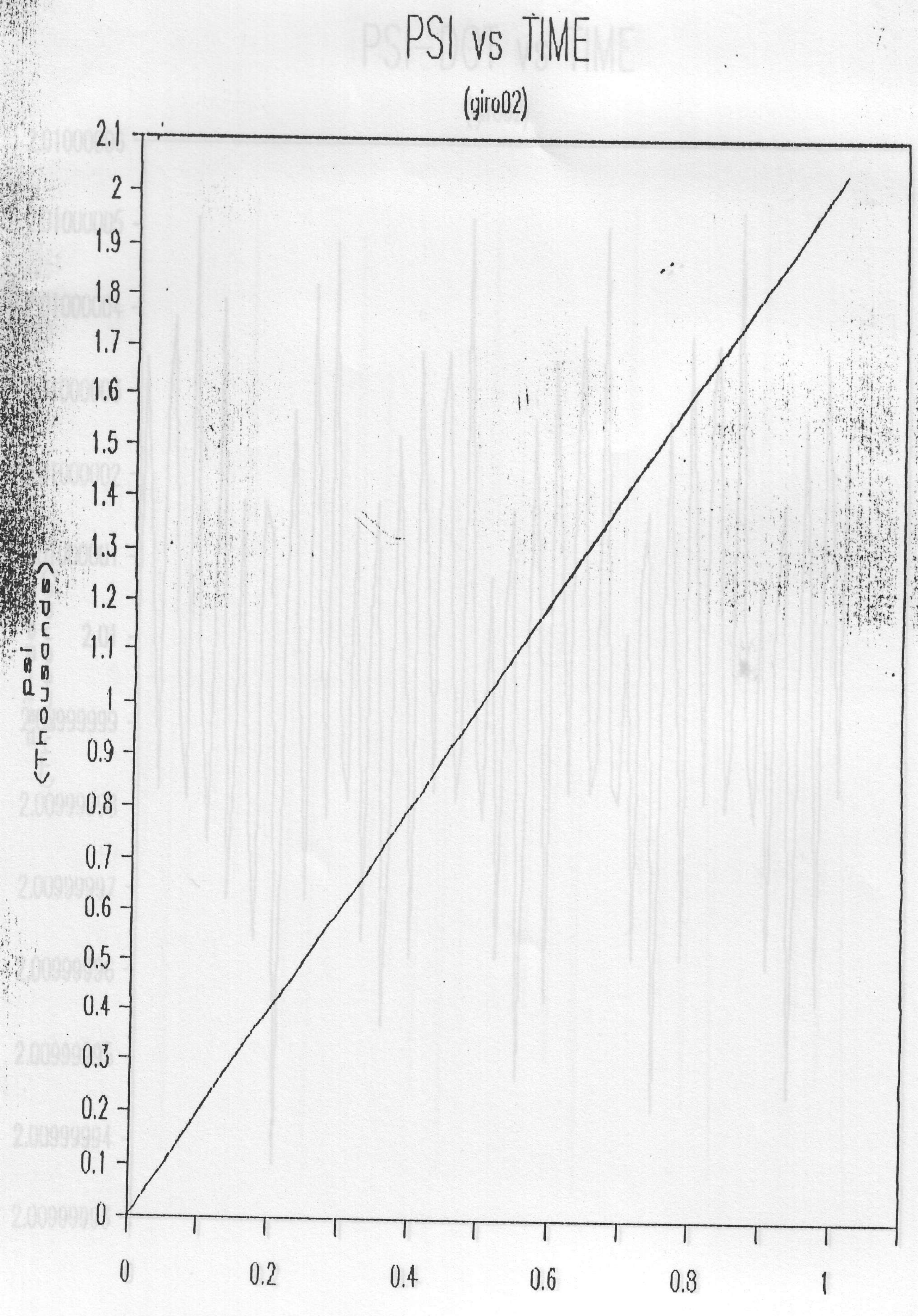

FI / FI-DOT
(giro02)
fi-dot
(Thousands)
50
40
30
20
10
0
-10
-20
-30
-40
-25
-15
-5
5
15
25
fi
A = B = C = D = 0


PSI / PSI-DOT
(giro02)
2.01000006
2.01000005
2.01000004
2.01000003
2.01000002
2.01000001
2.01
2.00999999
2.00999998
2.00999997
2.00999996
2.00999995
2.00999994
2.00999993
psi-dot (Thousands)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
(Thousands)
psi
— A = B = C = D = 0

THETA/ THETA-DOT

(giro02)

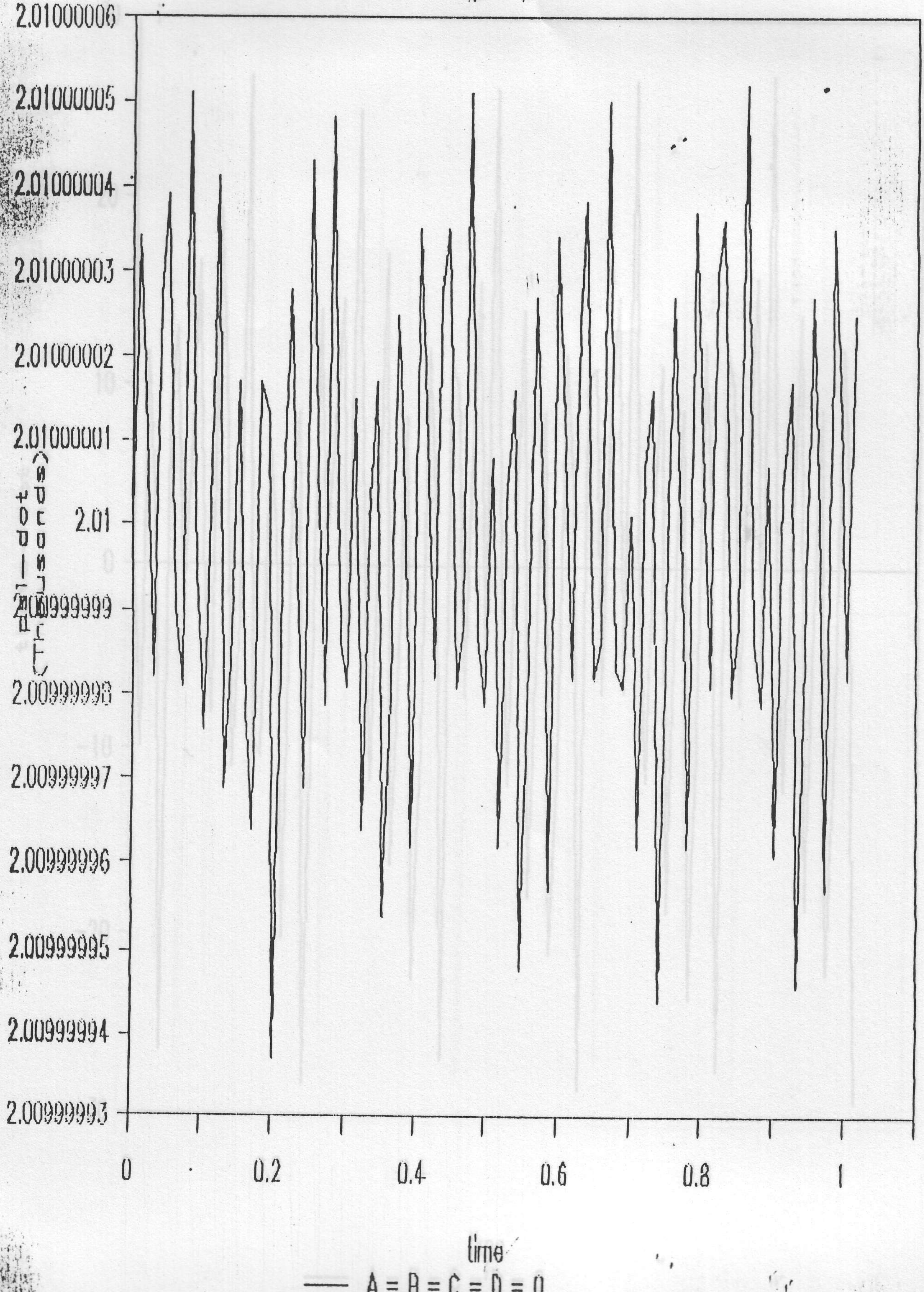




PSI vs TIME
(giro02)
2.1
2
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
psi
(Thousands)
0
0.2
0.4
0.6
0.8
1
time
A = B = C = D = 0

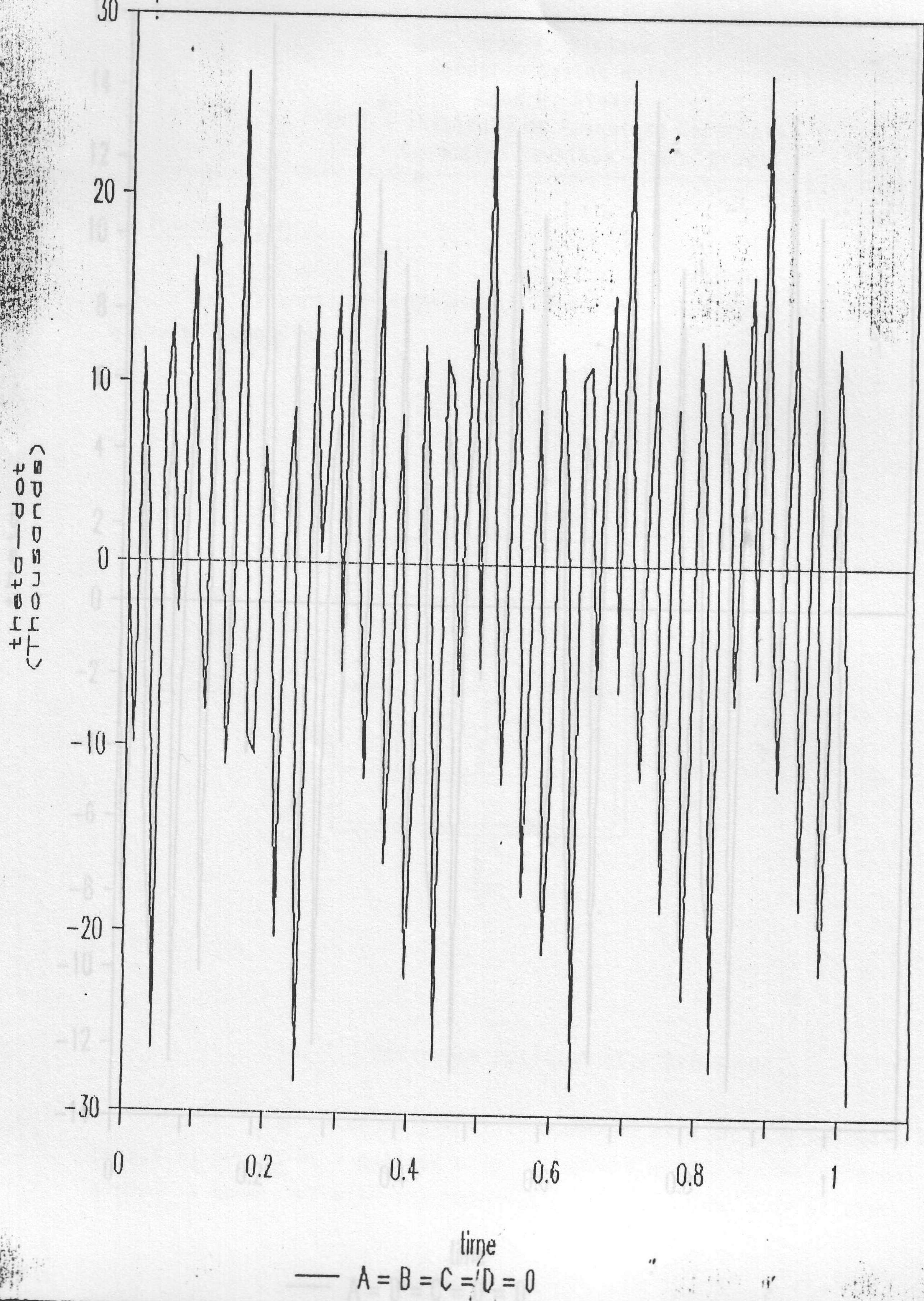
PSI-DOT vs TIME

(giro02)



THETA-DOT vs TIME

(giro02)



THETA vs TIME
(giro02)