DYNAMICS DAYS SOUTH AMERICA 2010
International Conference on Chaos and Nonlinear Dynamics

**INPE – National Institute for Space Research**
**São José dos Campos – SP – Brazil – July 26-30, 2010**

# Estimating hyperbolicity of chaotic bidimensional maps

*Cesar Manchein*[1,2], *Matteo Sala*[2], *Roberto Artuso*[2]

[1]Departamento de Física, Universidade Federal do Paraná, 81531-980 Curitiba, PR, Brazil, cmanchein@gmail.com
[2]Center for Nonlinear and Complex Systems and Dipartamento di Fisica e Matematica, Università degli Sudi dell'Insubria, Via Valleggio 11, 22100 Como, Italy, matteo.sala@uninsubria.it, roberto.artuso@uninsubria.it

**keywords:** sticky motion, covariant Lyapunov vectors, hyperbolicity.

## 1. PLAN AND MOTIVATION

In this work we applied to bidimensional chaotic maps the numerical method proposed by Ginelli *et al.* [1] that allows to calculate in each point of an orbit the vectors *tangent* to the (stable/unstable) invariant manifolds of the system, i.e. the so called *covariant Lyapunov vectors* (CLV); through this knowledge it is possible to calculate the transversal angle between the manifolds and, through statistics, quantify the degree of hyperbolicity. We started by testing the Hénon and Lozi attractors for specific values of parameters, and then focused on hyperbolicity of the Chirikov-Taylor map (standard map) for parameter values such that islands exist; in their neighborhood chaotic orbits experience transients of regular motion, i.e. *stickiness* [2], that is known to be directly associated with tangencies (i.e. zero angle) between the manifolds. Hence statistics of angles can be also used in conservative systems to test whether an orbit is sticky or not.

## 2. THE COVARIANT LYAPUNOV VECTORS (CLV)

By definition, hyperbolic dynamical systems are those for which in (almost) every point of its phase space no tangency is present, i.e. the stable and unstable manifolds are everywhere transversal. Knowing the directions $\mathbf{v}_n^u$ , $\mathbf{v}_n^s$ respectively tangent to the unstable and stable manifold in the orbit point $\mathbf{x}_n$, one can calculate the angle between manifolds by the formula

$$\phi_n = \cos^{-1} |\mathbf{v}_n^u \cdot \mathbf{v}_n^s| \ \in \ [0, \tfrac{\pi}{2}] \tag{1}$$

with absolute value since hyperbolicity only requires transversality ($\phi_n \neq 0$).

## 3. NUMERICAL RESULTS

### 3.1. The Hénon and Lozi Maps

The Hénon and Lozi maps are discrete-time dynamical systems defined respectively by the two-times evolutions :

$$\textbf{Hénon map:} \quad x_{n+1} = 1 - a x_n^2 + b x_{n-1}; \tag{2}$$

$$\textbf{Lozi map:} \quad x_{n+1} = 1 - a|x_n| + b x_{n-1}. \tag{3}$$

They are indeed bidimensional systems and their phase space can be considered as the plane $(x_n, x_{n-1})$.
We studied both of them at fixed parameter values $a = 1.4$ and $b = 0.3$, so each one is a *dissipative* system for which a strange attractor does exist. Since forward iteration of these maps brings orbits to evolve either to infinity or over the attractor depending on initial conditions, the only points from which is possible to localize information are those belonging to the attractors.

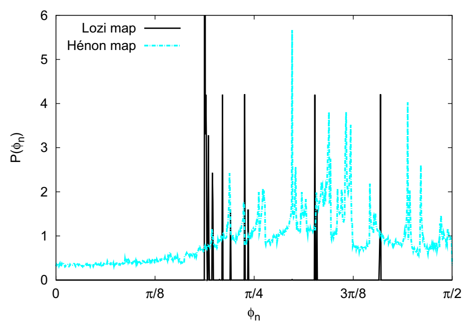


**Figura 1 – Probability distribution of angles between stable and unstable manifolds for the Hénon map (cyan line) and the Lozi map (black line). Both curves were rescaled by a factor $10^2$ and $10^3$, respectively.**

Figure 1 shows the behavior of angles distributions: for the Hénon map (cyan line) angles ranges completely over $[0, \frac{\pi}{2}]$ and arbitrarily small values are found, while in the Lozi map case (black line) the distribution is a sum of delta functions and is bounded away from zero. We remark that both results are consistent with the ones found in literature, from which is known that only the Lozi map is (non-uniformly) hyperbolic.

### *3.2. Standard Map*

The standard map, being the paradigm of all chaotic Hamiltonian maps, is the ideal model to study hyperbolicity in conservative systems. In canonical coordinates it has the form :

$$\begin{aligned} p_{n+1} &= p_n + K\sin(x_n) \\ x_{n+1} &= x_n + p_{n+1} \mod 2\pi, \end{aligned} \tag{4}$$

where $K$ is the nonlinear parameter. When the islands are present, chaotic trajectories can become temporarily trapped in the complicated neighborhood of the islands-around-islands (quasi-fractal) structure, undergoing a "sticky" transient around resonances.

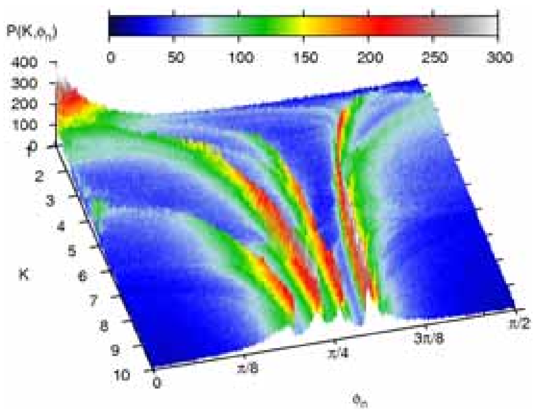


**Figura 2 – Probability distributions $P(K, \phi_n)$ of angles between stable and unstable manifolds for a scanning over nonlinear parameter $K$; calculated over 10 trajectories for each value of $K$, up to $n = 10^4$ iterations. With increasing $P(\phi_n, K)$ color changes from light to dark.**

Figure 2 shows the angles distributions $P(\phi_n, K)$ for a scan of 501 values of $K$ equally spaced over the range $[1, 10]$; each distribution was created using 10 initial conditions iterated $10^4$ times. In parallel we observed the phase space distributions of the transversal angles (see Figs. 3 and 4 (left and right)), i.e. we associated a color to every angle value and plot it in each orbit point. Since chaotic trajectories almost fill out areas, these plots show spatial variation of hyperbolicity. Until parameter $K$ is kept less than the value 4 the tails toward zero of the distributions are nonzero, meaning that lots of tangencies are present associated to islands neighborhoods i.e. sticky motion; for bigger $K$ tails drop down and one can observe the formation of sharp peaks. This due to the fact that for higher $K$ manifolds are mainly transversal i.e. tangencies occupy small areas and the angles at which manifold traverse are more similar (on average) leading to accumulations in their distributions.

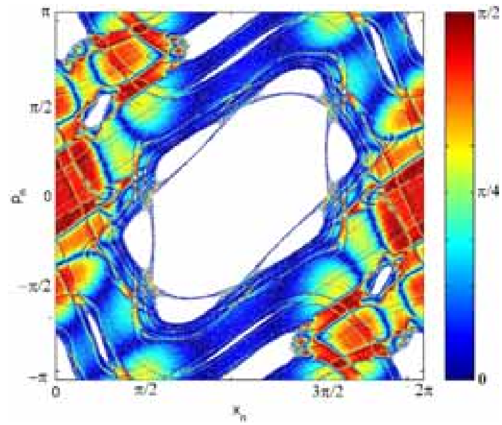

**Figura 3 – Standard map, $K = 1.5$; in color, $\phi_n$ transversal angle (rad, red=$\frac{\pi}{2}$, blue=0).**

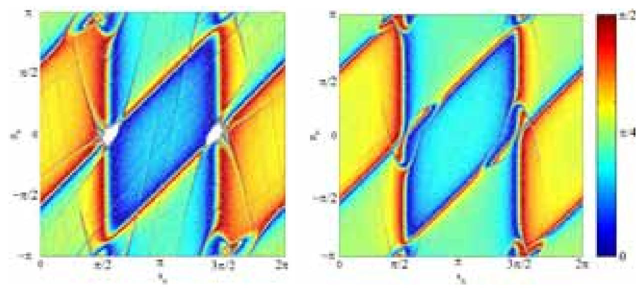

**Figura 4 – Standard map, $K = 6.5$ (left) and $K = 10$ (right); in color, $\phi_n$ transversal angle (rad, red=$\frac{\pi}{2}$, blue=0).**

## 4. CONCLUSIONS AND ACKNOWLEDGMENTS

In this numerical work we investigated the behavior of transversal angles of invariant manifolds for dissipative (Hénon and Lozi maps) and conservative (standard map) systems; the procedures we employed were based on the algorithm proposed by Ginelli *et al.* [1], although slightly different since we optimized them for bidimensional maps avoiding the direct use of matrices in computation (M. Sala, R. Artuso, work in preparation). The results were completely in line with the existing knowledge about hyperbolicity of the studied systems (i.e. non-uniform hyperbolicity for the Lozi map, general non-hyperbolicity for the Hénon and standard maps), emphasizing the complex structures of chaotic and sticky orbits in conservative systems.

C.M. thanks CNPq Brazil for financial support.

### Referências

[1] F. Ginelli, P. Poggi, A. Turchi, H. Chaté, R. Livi, and A. Politi, "Characterising dynamics with covariant Lyapunov vectors," Phys. Rev. Lett. Vol. 99, pp. 130601-130604, 2007.

[2] G. M. Zaslavsky, "Hamiltonian Chaos & Fractional Dynamics." New York: Oxford University Press, 2008.