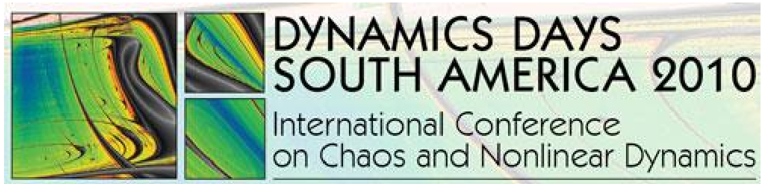

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ASTEROSEISMOLOGY OF RAPIDLY ROTATING STARS AND OPTICAL BILLIARDS

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In [**?**], Lignières and Georget present an asymptotic analysis of high-frequency acoustic modes in rapidly rotating stars, based on acoustic ray dynamics. They remark that, as the star rotates, its boundary is a deformation of a sphere on the equatorial plan, i.e. it is a 4-periodic symmetric ovoid. Then they show that, using the symetries of rotation, the acoustic rays can be described in two ways: either as trajectories of a particle under a classical 2-degree of freedom Hamiltonian depending on the frequence of rotation of the star, with a potential tending to infinity when approaching the boundary of the star; or as trajectories of optical rays in an isotropic 2-dimensional medium, with medium index depending on the distance to the center of the star.

This analysis produces a new kind of billiard problem, called optical billiard, where the optical ray travels along the geodesics of a riemannian metric associated to the medium index $n(x, y)$ inside the boundary curve, performing elastic reflections on the impacts with the boundary, where elastic means angle of incidence equal angle of reflection when measured by the internal product induced by the riemannian metric. Taking as coordinates a parameter for the boundary curve and the angle of reflection, the billiard problem defines a billiard map.

Integrating numerically the equations of motion, for a given frequency of rotation, Lignières and Georget obtain the picture bellow (originally on [**?**]), where they observe, on the phase space, the coexistence of elliptic islands and "chaotic" seas surrounding them:

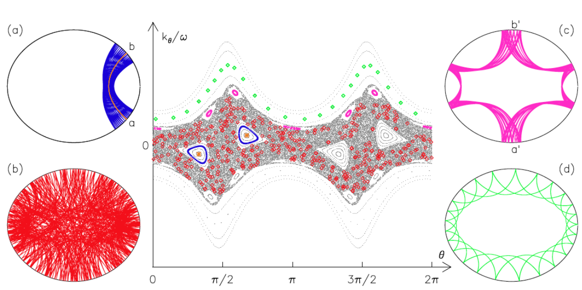

Figure 1 –

This coexistence is extremely important for the determination of two subsets of frequency modes: a regular one, corresponding to the islands, and a irregular one, corresponding to the "chaotic" trajectories, that will enable them to describe the pulsation of the star.

When the medium index $n(x, y) \equiv 1$, we recover the traditional billiard map, with straight trajectories. In [**?**], together with M.J.Dias Carneiro and S.Oliffson Kamphorst, we have proved that, in this traditional case and for convex billiard tables, the generic dynamical picture is: rotational invariant curves, containing instability zones between them. On each instability zone, there is the coexistence of a countable number of periodic islands (containing the regular motion) and an instability set, that corresponds to the closure of the stable (or the unstable) curves of a hyperbolic periodic orbit (where the motion seems numerically "chaotic"), as can be seen on the example pictured bellow.

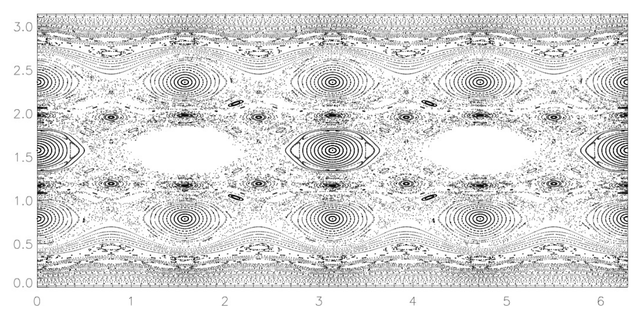

Figure 2 – Phase space of a classical billiard map on a 4-periodic symmetric table.

The main question of this work is: taking a medium index n depending only on the distance to the center of the billiard table, the generic optical billiard map has the same dynamical features as for the traditional case?

In order to answer this question, we begin our work by looking at the circular billiard. We suppose that , for polar coordinates (r, θ), we have a continuous and strictly positive function $n = n(r)$ defining the medium index. The optical Lagrangian is then given by

$$L(r, \theta, \dot{r}, \dot{\theta}) = n^2(r)\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

and the geodesics are the solutions of

$$\begin{aligned}\ddot{r} &= -\frac{1}{n}\frac{dn}{dr}\dot{r}^2 + \left(\frac{1}{n}\frac{dn}{dr} + \frac{1}{r}\right)(r\dot{\theta})^2 \\ \ddot{\theta} &= \left(-\frac{1}{n}\frac{dn}{dr} + \frac{1}{r}\right)\dot{r}\dot{\theta}\end{aligned}$$

We have then that the straight lines passing by the origin are geodesics and the geodesic circles centered at the origin are euclidean circles. It follows also that euclidean circles centered at the origin are geodesics if and only if $n(r) = ae^{-r}$.

Since the equations of the geodesics do not depend on θ it is easy to show that if $\gamma(t) = (r(t), \theta(t))$ is a geodesic then $\gamma_\alpha(t) = (r(t), \theta(t) + \alpha)$ and $\gamma_-(t) = (r(t), -\theta(t))$ are also geodesics, for any angle α.

This implies that the geodesic triangles with one vertice at the origin and the others in a geodesic circle centered at the origin have equal base angles.

Given then a continuous and strictly positive medium index $n = n(r)$, let V be a normal neighbourhood of the origin for the associated riemannian metric $g(r, \theta) = n^2(r)\left(dr^2 + r^2 d\theta^2\right)$. Let C be a geodesic circle centered at the origin and contained on V. The optical billiard map is then well defined on C and we can prove that:

Theorem: The optical billiard map on the geodesic circle C is integrable.

References

[] F.Lignières, B.Georgeot: Asymptotic analysis of high-frequency acoustic modes in rapidly rotating stars, A&A 500, 1173Ð1192 (2009) DOI: 10.1051/0004-6361/200811165

[] M.J.Dias Carneiro, S.Oliffson Kamphorst, S.Pinto-de-Carvalho: Periodic orbits of generic oval billiards, Nonlinearity 20 24532462 (2007)

[] M.P. do Carmo: Geometria Diferencial de Curvas e Superfícies, SBM, 2005.