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Tachyacoustic cosmology: An alternative to inflation

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We consider an alternative to inflation for the generation of superhorizon perturbations in the Universe in which the speed of sound is faster than the speed of light. We label such cosmologies, first proposed by Armendariz-Picon, *tachyacoustic*, and explicitly construct examples of noncanonical Lagrangians which have superluminal sound speed, but which are causally self-consistent. Such models possess two horizons, a Hubble horizon and an acoustic horizon, which have independent dynamics. Even in a decelerating (noninflationary) background, a nearly scale-invariant spectrum of perturbations can be generated by quantum perturbations redshifted outside of a shrinking acoustic horizon. The acoustic horizon can be large or even infinite at early times, solving the cosmological horizon problem without inflation. These models do not, however, dynamically solve the cosmological flatness problem, which must be imposed as a boundary condition. Gravitational wave modes, which are produced by quantum fluctuations exiting the Hubble horizon, are not produced.

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I. INTRODUCTION: WHY INFLATION WORKS

Inflationary cosmology [1–3] is the most successful and widely accepted paradigm for understanding the very early Universe. By all measures inflation is a compelling and scientifically useful theory, and makes quantitative predictions which have been strongly supported by recent data [4–6]. The two main hallmarks of inflationary cosmology are solutions to the flatness and horizon problems of the standard big bang cosmology: why is the Universe so close to geometrically flat, and how did the apparent acausal structure of the Universe arise? ‘‘Acausal’’ more specifically means that the Universe is approximately homogeneous on scales larger than a Hubble length H^{-1}, and in addition exhibits a spectrum of density perturbations which is correlated on scales larger than a Hubble length. Such *superhorizon* correlations are generated in inflation by accelerated expansion, $\ddot{a}/a > 0$, where $a(t)$ is the cosmological scale factor, which means that the comoving Hubble length $d_H \simeq (aH)^{-1}$ shrinks with the expansion of the Universe,

$$\frac{d}{d\ln a}(aH)^{-1} < 0. \tag{1}$$

Therefore, quantum perturbations, which have constant wavelength in comoving units, are smaller than the Hubble length at early times, and are redshifted to larger than the Hubble length at late times, where they are ‘‘frozen’’ as classical perturbations. Furthermore, as long as the Hubble constant H is slowly varying with time, the perturbations generated in inflation are nearly scale-invariant, consistent with observation. Furthermore, the solution to the horizon problem and the flatness problem are linked in inflation via a conservation law,

$$\frac{d}{d\ln a}\frac{|\Omega - 1|}{d_H^2} = 0. \tag{2}$$

Through this conservation law, a universe with shrinking comoving horizon size is identical to a universe which is evolving toward flatness,

$$\frac{d}{d\ln a}|\Omega - 1| < 0. \tag{3}$$

Inflation therefore solves the horizon and flatness problems of the standard big bang with a *single* mechanism: accelerated expansion.

However, inflation is not the only way to accomplish this goal, as can be seen from the fact that the acceleration $\ddot{a}$ appears nowhere in the conservation law (2). To solve both the horizon and flatness problems, it is sufficient to have a shrinking comoving Hubble length. One way to do this is accelerated expansion, but another is to have a collapsing universe, $H = (\dot{a}/a) < 0$. A collapsing, matter- or radiation-dominated universe also has a shrinking comoving Hubble length, which will generate perturbations in a manner similar to inflation. This is the mechanism used by the Ekpyrotic scenario [7] to construct a cosmology consistent with observations. It is also possible to decouple the horizon and flatness problems, for example, in theories with a varying speed of light, so that the causal horizon is much larger than the Hubble length [8]. Such theories can in principle solve the horizon problem, but not the flatness problem, since the conservation law (2) is violated. It is also possible to solve the horizon problem by a

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universe which is much older than a Hubble time as in string gas cosmology [9] or island cosmology [10,11], or by the inclusion of extra dimensions [12,13]. However, it has been argued that inflation and ekpyrosis are the *only* mechanisms for generating a scale-invariant spectrum of perturbations [7,14].

In this paper, we discuss a method of solving the cosmological horizon problem and seeding scale-invariant primordial perturbations in a cosmology with decelerating expansion and a corresponding *growing* comoving Hubble horizon. The key to implementing such a model is the fact that curvature perturbations are not generated at the Hubble horizon, but at the acoustic horizon determined by the speed of sound of a scalar field. For canonical field theories, the two are identical, but for noncanonical field theories, they are not. If one has a decaying, superluminal sound speed, curvature perturbations can be generated outside the Hubble horizon without inflation. We propose the term *tachyacoustic* for such cosmologies, which are closely related to varying speed of light theories. This idea has some history: such cosmologies were first proposed by Armendariz-Picon in the context of modified dispersion relations [15], and the generation of perturbations in such cosmologies was further considered by Piao [16]. The idea reemerged in the context of varying speed of light theories by Magueijo [17], and noncanonical Lagrangians by Magueijo [18] and Piao [19]. In this paper, we outline a general approach to such cosmologies based on the generalization of the inflationary flow formalism [20] introduced by Bean, *et al.* for the case of arbitrary Lagrangians [21]. We find that there is a class of Lagrangians with the necessary properties for tachyacoustic cosmologies, and discuss two interesting examples. We find that it is straightforward to generate nearly scale-invariant perturbations for these Lagrangians, and show that they have the property of reducing to instantonlike solutions with infinite sound speed on the initial-time boundary of the spacetime. We speculate that this property may allow a self-consistent description of tachyacoustic cosmologies within a Wheeler-DeWitt description of quantum cosmology. Finally, we show that such models are causally self-consistent, and argue that they form a viable class of alternatives to inflation.

II. TACHYACOUSTIC COSMOLOGY

In this paper, we consider a way of generating scale-invariant superhorizon cosmological perturbations based on noncanonical scalar-field Lagrangians with a speed of sound faster than the speed of light, $c_S > 1$. If the Universe is dominated by a scalar field with speed of sound c_S, the relevant horizon for the generation of density perturbations is not the Hubble horizon $d_H \simeq (aH)^{-1}$ but the acoustic horizon,

$$D_H \simeq \frac{c_S}{aH}. \tag{4}$$

Mode freezing at the acoustic horizon is well known in noncanonical inflation models, for example, k-Inflation [22] and Dirac-Born-Infeld (DBI) inflation [23]. In noncanonical inflation models, the Hubble horizon and the acoustic horizon are *both* shrinking in comoving units, resulting in the generation of density perturbations at the acoustic horizon and gravitational wave perturbations at the Hubble horizon [24]. However, the comoving Hubble horizon need not be shrinking to generate curvature perturbations: all that is required is that the *acoustic* horizon be shrinking, $dD_H/d\ln a < 0$. In this case, if curvature perturbations are to be generated on scales larger than the Hubble horizon, it is necessary that the acoustic horizon be larger than the Hubble horizon, which requires a speed of sound greater than the speed of light. Such theories were studied recently by Babichev *et al.* [25,26], who showed that k-essence theories with $c_S > 1$ are causally self-consistent (see the Appendix), and can be mapped to bimetric theories with two "light cones," one given by the Hubble horizon, and the other given by the acoustic horizon, which can be larger than the Hubble horizon without the presence of closed timelike loops. This opens the possibility that one can construct a decelerating cosmology which nonetheless generates perturbations on super-Hubble scales via a superluminal, shrinking acoustic cone.

To explicitly construct such a model, consider a DBI Lagrangian,

$$\mathcal{L}(\phi, X) = -f^{-1}(\phi)\sqrt{1 - f(\phi)\dot{\phi}^2} + f^{-1}(\phi) - V(\phi), \tag{5}$$

where we take the metric to have negative signature $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, so that the kinetic term for the field is positive,

$$\dot{\phi}^2 \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi > 0. \tag{6}$$

Such Lagrangians often arise in string theory, for example, in the case of DBI inflation. For the moment, we will not attempt to make a connection with string theory, but will take the Lagrangian (5) as a phenomenological *ansatz*. Take the background spacetime to be a flat, Friedmann-Robertson-Walker (FRW) space, $g_{\mu\nu} = a^2\eta_{\mu\nu}$, so that the scale factor evolves as

$$a \propto \exp\left(\int H dt\right) \propto e^{-N}, \tag{7}$$

where we define the number of e-folds N as[1]

$$N \equiv -\int H dt. \tag{8}$$

There is a class of exact solutions [27,28] to the equation of

[1] We use the usual convention that $N \to \infty$ corresponds to early time, and $N \to -\infty$ corresponds to late time.

motion for the field ϕ characterized by two dimensionless flow parameters ϵ and s, where

$$\epsilon \equiv \frac{1}{H}\frac{dH}{dN} = \text{const}, \tag{9}$$

and

$$s \equiv -\frac{1}{c_S}\frac{dc_S}{dN} = \text{const}. \tag{10}$$

The parameter ϵ has its usual interpretation in terms of the equation of state of the scalar field,

$$p = \rho\left(\frac{2}{3}\epsilon - 1\right). \tag{11}$$

For $\epsilon = \text{const}$, the scale factor evolves as a power law, $a \propto t^{1/\epsilon}$, so that the expansion is accelerating (i.e. inflation) for $\epsilon < 1$. The speed of sound evolves as

$$c_S = \sqrt{1 - f(\phi)\dot{\phi}^2} \propto e^{-sN}, \tag{12}$$

and the Hubble parameter evolves as

$$H = \frac{\dot{a}}{a} \propto e^{\epsilon N}. \tag{13}$$

The parameter ϵ is a positive-definite quantity for $p \geq -\rho$, so that the Hubble constant always decreases with expansion. In contrast, the parameter s can take either sign, with $s > 0$ corresponding to a sound speed which increases with expansion, and $s < 0$ corresponding to an decreasing sound speed. (See Ref. [28] for a detailed derivation of this solution.) The important dynamics for the generation of perturbations is the time evolution of the corresponding horizons in comoving units. The comoving Hubble horizon evolves as

$$d_H \propto (aH)^{-1} \propto e^{(1-\epsilon)N} \propto \tau, \tag{14}$$

where τ is the conformal time. The Hubble horizon is shrinking in comoving units for $\epsilon < 1$, which is identical to accelerated expansion, and is the usual condition for inflation. The acoustic horizon behaves as

$$D_H \propto \frac{c_S}{aH} \propto e^{(1-\epsilon-s)N} \propto \tau^{(1-\epsilon-s)/(1-\epsilon)}. \tag{15}$$

Therefore the condition for a shrinking acoustic horizon, $1 - \epsilon - s > 0$, is *not* identical to accelerated expansion. For $\epsilon > 1$ and $s < 1 - \epsilon$, the expansion is noninflationary, the Hubble horizon is growing in comoving units, and the acoustic horizon is shrinking. The initial singularity is at $\tau = 0$, and we see immediately that for the tachyacoustic solution, the speed of sound in the scalar field is *infinite* at the initial singularity, and the acoustic horizon is likewise infinite in size. Therefore, such a cosmology presents no "horizon problem" in the usual sense, since even a spatially infinite spacetime is causally connected on the initial-time boundary. Furthermore, unlike in the case of inflation, there is no period of reheating necessary, since the cosmological evolution can be radiation-dominated throughout and the cosmic temperature is not driven exponentially to zero.

In the next section, we use the generalized flow function approach of Bean, *et al.* [21] to construct a class of Lagrangians with solutions of the type outlined above, with constant flow parameters. In these solutions, the scale factor evolves as a power law in time and the equation of motion for curvature perturbations can be solved exactly, which we discuss in Sec. VI.

III. FLOW HIERARCHY FOR GENERAL K-ESSENCE MODELS

We now generalize the discussion in the last section to an arbitrary k-essence model. Consider a general Lagrangian of the form $\mathcal{L} = \mathcal{L}[X, \phi]$, where $2X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the canonical kinetic term ($X > 0$ according to our choice of the metric signature). The energy density ρ and pressure p are given by

$$p = \mathcal{L}(X, \phi), \tag{16}$$

$$\rho = 2X\mathcal{L}_X - \mathcal{L}. \tag{17}$$

The speed of sound is given by

$$c_S^2 \equiv \frac{p_X}{\rho_X} = \left(1 + 2X\frac{\mathcal{L}_{XX}}{\mathcal{L}_X}\right)^{-1}, \tag{18}$$

where the subscript "X" indicates a derivative with respect to the kinetic term. Throughout this section, unless otherwise stated, we will follow closely Bean *et al.* [21]. We define the first three *flow parameters* as derivatives with respect to the number of e-folds, $dN = -Hdt$[2]:

$$\epsilon \equiv \frac{1}{H}\frac{dH}{dN}, \tag{19}$$

$$s \equiv -\frac{1}{c_S}\frac{dc_S}{dN}, \tag{20}$$

$$\tilde{s} \equiv \frac{1}{\mathcal{L}_X}\frac{d\mathcal{L}_X}{dN}. \tag{21}$$

The Friedmann equation can be written in terms of the reduced Planck mass $M_P = 1/\sqrt{8\pi G}$

$$H^2 = \frac{1}{3M_P^2}\rho = \frac{1}{3M_P^2}(2X\mathcal{L}_X - \mathcal{L}), \tag{22}$$

and the continuity equation is

$$\dot{\rho} = 2H\dot{H} = -3H(\rho + p) = -6HX\mathcal{L}_X. \tag{23}$$

For monotonic field evolution, the field value ϕ can be used as a "clock," and all other quantities expressed as

[2]The parameters s and $\tilde{s}$ correspond to the parameters κ and $\tilde{\kappa}$ in Bean, *et al.* [21].

functions of ϕ, for example $X = X(\phi)$, $\mathcal{L} = \mathcal{L}[X(\phi), \phi]$, and so on. We consider the homogeneous case, so that $\dot{\phi} = \sqrt{2X}$. Using

$$\frac{d}{dt} = \dot{\phi}\frac{d}{d\phi} = \sqrt{2X}\frac{d}{d\phi}, \tag{24}$$

we can rewrite the Friedmann and continuity equations as the *Hamilton-Jacobi* equations,

$$\dot{\phi} = \sqrt{2X} = -\frac{2M_P^2}{\mathcal{L}_X}H'(\phi), \tag{25}$$

$$3M_P^2H^2(\phi) = \frac{4M_P^4H'(\phi)^2}{\mathcal{L}_X} - \mathcal{L}, \tag{26}$$

where a prime denotes a derivative with respect to the field ϕ. The number of e-folds dN can similarly be rewritten in terms of $d\phi$ by

$$dN \equiv -Hdt = -\frac{H}{\sqrt{2X}}d\phi \tag{27}$$

$$= \frac{\mathcal{L}_X}{2M_P^2}\left(\frac{H(\phi)}{H'(\phi)}\right)d\phi. \tag{28}$$

The flow parameters ϵ, s, and $\tilde{s}$ (19) can therefore be written as derivatives with respect to the field ϕ as

$$\epsilon(\phi) = \frac{2M_P^2}{\mathcal{L}_X}\left(\frac{H'(\phi)}{H(\phi)}\right)^2, \tag{29}$$

$$s(\phi) = -\frac{2M_P^2}{\mathcal{L}_X}\frac{H'(\phi)}{H(\phi)}\frac{c_S'(\phi)}{c_S(\phi)}, \tag{30}$$

$$\tilde{s}(\phi) = \frac{2M_P^2}{\mathcal{L}_X}\frac{H'(\phi)}{H(\phi)}\frac{\mathcal{L}'_X}{\mathcal{L}_X}. \tag{31}$$

Taking successive derivatives d/dN with respect to the number of e-folds yields an infinite hierarchy of flow equations [20,21],

$$\begin{gathered}
\frac{d\epsilon}{dN} = -\epsilon(2\epsilon - 2\tilde{\eta} + \tilde{s}), \qquad \frac{d\tilde{\eta}}{dN} = -\tilde{\eta}(\epsilon + \tilde{s}) + {}^2\lambda, \qquad \frac{ds}{dN} = -s(\epsilon - \tilde{\eta} + \tilde{s} + s) + \epsilon\rho, \\
\frac{d\tilde{s}}{dN} = -\tilde{s}(\epsilon - \tilde{\eta} + 2\tilde{s}) + \epsilon\, {}^1\beta, \qquad \frac{d\,{}^\ell\lambda}{dN} = -{}^\ell\lambda[\ell(\tilde{s} + \epsilon) - (\ell - 1)\tilde{\eta}] + {}^{\ell+1}\lambda, \\
\frac{d\,{}^\ell\alpha}{dN} = -{}^\ell\alpha[(\ell - 1)(\epsilon - \tilde{\eta}) + \ell\tilde{s} + s] + {}^{\ell+1}\alpha, \qquad \frac{d\,{}^\ell\beta}{dN} = -{}^\ell\beta[(\ell - 1)(\epsilon - \tilde{\eta}) + (\ell + 1)\tilde{s}] + {}^{\ell+1}\beta,
\end{gathered} \tag{32}$$

where the higher-order flow parameters are defined as follows, where $\ell = 1, \ldots \infty$ is an integer parameter:

$$\begin{aligned}
\tilde{\eta}(\phi) &= {}^1\lambda = \frac{2M_P^2}{\mathcal{L}_X}\frac{H''(\phi)}{H(\phi)}, \\
{}^\ell\lambda(\phi) &= \left(\frac{2M_P^2}{\mathcal{L}_X}\right)^\ell\left(\frac{H'(\phi)}{H(\phi)}\right)^{\ell-1}\frac{1}{H(\phi)}\frac{d^{\ell+1}}{d\phi^{\ell+1}}H(\phi), \\
{}^\ell\alpha(\phi) &= \left(\frac{2M_P^2}{\mathcal{L}_X}\right)^\ell\left(\frac{H'(\phi)}{H(\phi)}\right)^{\ell-1}\frac{1}{c_S^{-1}(\phi)}\frac{d^{\ell+1}}{d\phi^{\ell+1}}c_S^{-1}(\phi), \\
{}^\ell\beta(\phi) &= \left(\frac{2M_P^2}{\mathcal{L}_X}\right)^\ell\left(\frac{H'(\phi)}{H(\phi)}\right)^{\ell-1}\frac{1}{\mathcal{L}_X}\frac{d^{\ell+1}}{d\phi^{\ell+1}}\mathcal{L}_X.
\end{aligned} \tag{33}$$

Solutions to this infinite hierarchy of flow equations are equivalent to solutions of the scalar-field equation of motion. In the next section, we specialize to the case where the flow parameters are constant, which results in an exactly solvable system.

IV. COSMOLOGICAL SOLUTIONS FOR CONSTANT FLOW PARAMETERS

The simplest way to solve the flow equations derived in the preceding section is to take all of the flow parameters to be constant,

$$\frac{d\epsilon}{dN} = \frac{ds}{dN} = \frac{d\tilde{s}}{dN} = \frac{d\,{}^\ell\lambda}{dN} = \frac{d\,{}^\ell\alpha}{dN} = \frac{d\,{}^\ell\beta}{dN} = 0. \tag{34}$$

Then, from (19)–(21) we easily find the following relations:

$$H \propto e^{\epsilon N}, \qquad c_S \propto e^{-sN}, \qquad \mathcal{L}_X \propto e^{\tilde{s}N}. \tag{35}$$

The first two are identical to the DBI case, Eqs. (12) and (13), but in the fully general case $\mathcal{L}_X$ evolves independently of c_S. It is straightforward to verify that the full flow hierarchy (33) reduces to an exactly solvable set of algebraic equations, with the higher-order parameters expressed in terms of ϵ, s, and $\tilde{s}$. We can use the relations (29)–(31) to solve for $H(\phi)$, $c_S(\phi)$, and $\mathcal{L}_X(\phi)$ as follows: from Eqs. (29) and (31), we have

$$\tilde{s} = \frac{2M_P^2}{\mathcal{L}_X}\left(\frac{H'}{H}\right)\frac{\mathcal{L}'_X}{\mathcal{L}_X} = M_p\sqrt{2\epsilon}\frac{\mathcal{L}'_X}{\mathcal{L}_X} = \text{const.} \tag{36}$$

We then have a differential equation for $\mathcal{L}_X$,

$$\frac{\mathcal{L}'_X}{\mathcal{L}_X^{3/2}} = \frac{\tilde{s}}{M_P\sqrt{2\epsilon}} = \text{const}, \tag{37}$$

with solution

$$\mathcal{L}_X(\phi) = \frac{8\epsilon}{\tilde{s}^2}\left(\frac{M_P}{\phi}\right)^2, \tag{38}$$

where the integration constant has been absorbed into a field redefinition. From Eq. (35), the field then evolves as

$$\phi^2 \propto e^{-\tilde{s}N}, \tag{39}$$

so that the direction of the field evolution depends on the sign of $\tilde{s}$,

$$\frac{d\phi}{\phi} = -\frac{\tilde{s}}{2}dN. \tag{40}$$

Equation (29) then reduces to

$$\left(\frac{H'}{H}\right)^2 = \frac{4\epsilon^2}{\tilde{s}^2\phi^2}, \tag{41}$$

with solution

$$H \propto \phi^{\pm 2\epsilon/\tilde{s}}. \tag{42}$$

The sign ambiguity can be resolved by requiring that the Universe be expanding, $dH/dN > 0$, so that

$$H \propto \phi^{-2\epsilon/\tilde{s}} \propto e^{\epsilon N}. \tag{43}$$

Finally, we solve for the speed of sound using Eq. (30), which reduces to

$$\frac{c_S'}{c_S} = \frac{2s}{\tilde{s}} = \text{const}, \tag{44}$$

with solution

$$c_S \propto \phi^{2s/\tilde{s}}. \tag{45}$$

Since our choice of $N = 0$ corresponds to an arbitrary renormalization of the scale factor $a \propto e^{-N}$, we can without loss of generality define $c_S = 1$ at $N = 0$, so that the general solution for the background evolution is given by

$$\mathcal{L}_X = \frac{8\epsilon}{\tilde{s}^2}\left(\frac{M_P}{\phi}\right)^2, \tag{46}$$

$$H(\phi) = H_0\left(\frac{\phi}{\phi_0}\right)^{-2\epsilon/\tilde{s}}, \tag{47}$$

$$c_S(\phi) = \left(\frac{\phi}{\phi_0}\right)^{2s/\tilde{s}}, \tag{48}$$

where the field evolves as

$$\frac{\phi}{\phi_0} = e^{-\tilde{s}N/2}. \tag{49}$$

We can derive the time dependence of the scale factor using the Hamilton-Jacobi equation (25),

$$\dot{\phi} = \frac{\tilde{s}}{2}H(\phi)\phi = \sqrt{2X}, \tag{50}$$

so that the kinetic term can be written as

$$X(\phi) = \frac{\tilde{s}^2}{8}H^2(\phi)\phi^2. \tag{51}$$

Integrating expression (50) gives

$$H(t) = \frac{1}{\epsilon t}, \tag{52}$$

so that the scale factor evolves as a power law in time, consistent with the relation (9) between ϵ and the equation of state $w = p/\rho$,

$$a(t) \propto t^{1/\epsilon} = t^{2/3(1+w)}. \tag{53}$$

Radiation-dominated evolution therefore corresponds to $\epsilon = 2$, and matter-dominated evolution corresponds to $\epsilon = 3/2$. Inflation corresponds to $\epsilon < 1$. The comoving Hubble horizon evolves proportional the to the conformal time,

$$d_H \propto (aH)^{-1} \propto e^{(1-\epsilon)N} \propto \tau, \tag{54}$$

and the acoustic horizon evolves as

$$D_H \propto \frac{c_S}{aH} \propto e^{(1-\epsilon-s)N} \propto \tau^{(1-\epsilon-s)/(1-\epsilon)}, \tag{55}$$

identically to the DBI case discussed in Sec. II. For $\epsilon > 1$, the acoustic horizon is shrinking in comoving units for $s < 1 - \epsilon$. Note that this behavior is independent of the parameter $\tilde{s}$, which determines the form of the Lagrangian, as we discuss in the next section.

V. RECONSTRUCTING THE ACTION

In the past two sections we have solved the flow hierarchy for a model characterized by constant flow parameters, which allowed us to solve for $H(\phi)$, $c_S(\phi)$, and $\mathcal{L}_X(\phi)$; only the derivative of the Lagrangian with respect to the kinetic term X is determined. Therefore this solution corresponds not to a single action but a class of actions. In this section we derive a general equation for Lagrangians in this class, and discuss two specific examples.

From Eqs. (38) and (48), we see that the speed of sound c_S can be written in terms of $\mathcal{L}_X$

$$c_S^2 = C^{-1}\mathcal{L}_X^{-2s/\tilde{s}} = \left[1 + 2X\frac{\mathcal{L}_{XX}}{\mathcal{L}_X}\right]^{-1}, \tag{56}$$

where we have used Eq. (18), and defined

$$C \equiv \left(\frac{\tilde{s}^2\phi_0^2}{8M_P^2\epsilon}\right)^{2s/\tilde{s}}. \tag{57}$$

The result is a differential equation for the function $\mathcal{L}(X, \phi)$:

$$2X\mathcal{L}_{XX} + \mathcal{L}_X - C\mathcal{L}_X^n = 0, \tag{58}$$

where we have defined

$$n \equiv 1 + \frac{2s}{\tilde{s}}. \tag{59}$$

Therefore, by specifying a relationship between the parameters s and $\tilde{s}$, we can construct a Lagrangian as the solution to the differential equation (58). For example, a canonical Lagrangian with speed of sound $c_S = \text{const} = 1$ is just the case $s = 0$, so that $n = 1$ and $C = 1$, and Eq. (58) becomes

$$\mathcal{L}_{XX} = 0, \tag{60}$$

with general solution

$$\mathcal{L} = f(\phi)X - V(\phi). \tag{61}$$

Here $f(\phi)$ and $V(\phi)$ are free functions which arise from integration of the second-order equation (58). The function $f(\phi)$ can be eliminated by a field redefinition $d\varphi = \sqrt{f(\phi)}d\phi$, resulting in a manifestly canonical Lagrangian for φ, as we would expect from setting $c_S = 1$. We emphasize that Eq. (58) is constructed using the solution (48), and is not a general condition on the Lagrangian. That is, Eq. (58) allows us to construct a Lagrangian which admits solutions of the desired form, but those solutions are not necessarily unique. A canonical Lagrangian can support inflationary solutions, but not tachyacoustic solutions, and is therefore not of interest here. However, other choices of n do yield tachyacoustic solutions, and we focus on two such choices:

(1) $n = 0$: A cuscuton-like model.

(2) $n = 3$: A DBI model.

We discuss each case separately below.

A. n = 0: A cuscuton-like model

The case $n = 0$ corresponds to $\tilde{s} = -2s$ in (59), with solution

$$\mathcal{L}(X, \phi) = 2f(\phi)\sqrt{X} + CX - V(\phi). \tag{62}$$

This Lagrangian is similar to a "cuscuton" Lagrangian [29], with the addition of a term proportional to X. Unlike the original cuscuton model, which represents a causal field with infinite speed of sound, the solution obtained here is valid for the general case, in which the speed of sound can be finite. A similar cuscuton-like Lagrangian was considered in Ref. [19].

As in the canonical case, the functions $f(\phi)$ and $V(\phi)$ are free functions resulting from integrating Eq. (58). Unlike the canonical case, however, neither can be removed by a field redefinition. However, both functions are fully determined by our choice of solution with ϵ, s, and $\tilde{s}$ constant. Differentiating Eq. (62) with respect to X gives

$$\mathcal{L}_X = \frac{f(\phi)}{\sqrt{X}} + C = \frac{2\epsilon}{s^2}\left(\frac{M_P}{\phi}\right)^2, \tag{63}$$

where the right-hand side is the solution (46). Then

$$\begin{aligned} f(\phi) &= \sqrt{X}\left(\frac{2\epsilon}{s^2}\right)\left(\frac{M_P}{\phi_0}\right)^2\left[\left(\frac{\phi_0}{\phi}\right)^2 - 1\right] \\ &= \sqrt{X}\left(\frac{2\epsilon}{s^2}\right)\left(\frac{M_P}{\phi_0}\right)^2[c_S^2(\phi) - 1], \end{aligned} \tag{64}$$

where for $2\tilde{s} = -s$, the expression (48) for the speed of sound becomes

$$c_S(\phi) = \left(\frac{\phi_0}{\phi}\right). \tag{65}$$

The Lagrangian (62) can then be written as

$$\mathcal{L} = X\left(\frac{2\epsilon}{s^2}\right)\left(\frac{M_P}{\phi_0}\right)^2[2c_S^2(\phi) - 1] - V(\phi). \tag{66}$$

The Hubble parameter (47) is given by

$$H(\phi) = H_0\left(\frac{\phi}{\phi_0}\right)^{\epsilon/s}, \tag{67}$$

and we can then express the kinetic term as a function of ϕ using Eq. (51):

$$X(\phi) = \frac{s^2}{2}H^2\phi^2 = \frac{s^2}{2}\frac{\phi_0^2 H^2(\phi)}{c_s^2(\phi)}, \tag{68}$$

The Lagrangian (62) can then be written entirely as a function of the field ϕ,

$$\mathcal{L} = M_P^2\epsilon H^2(\phi)\left[2 - \frac{1}{c_S^2(\phi)}\right] - V(\phi). \tag{69}$$

The Hamilton-Jacobi equation (25) becomes

$$3M_P^2H^2 = 2M_P^2\epsilon H^2 - \mathcal{L} = V(\phi) + \frac{M_P^2\epsilon H^2}{c_S^2}, \tag{70}$$

and we have an expression for the potential $V(\phi)$,

$$V(\phi) = M_P^2H^2(\phi)\left[3 - \frac{\epsilon}{c_S^2(\phi)}\right]. \tag{71}$$

The Hubble parameter $H(\phi)$ and the speed of sound $c_S(\phi)$ are given by Eqs. (65) and (67), respectively. For $\phi/\phi_0 \ll 1$, the speed of sound is much greater than the speed of light, $c_S \gg 1$, and the potential is approximately

$$V(\phi) \simeq 3M_P^2H^2(\phi) = 3M_P^2H_0^2\left(\frac{\phi}{\phi_0}\right)^{2\epsilon/s}, \tag{72}$$

which can be recognized as a slow-roll-like solution dominated by the potential $H^2 \simeq V^2/3M_P^2$. For $s < 0$, the field is rolling away from the origin, and for $s < 1 - \epsilon$ the comoving acoustic horizon is shrinking and the solution is tachyacoustic.

B. n = 3: The DBI model

The case $n = 3$, corresponds to $\tilde{s} = s$; then, from (30) and (31), we find $\mathcal{L}_X = c_S^{-1}$. Equation (59) is then

$$c_S^2 = \frac{1}{C\mathcal{L}_X^2}, \tag{73}$$

so that we can take $C = 1$ without loss of generality. Therefore, the Lagrangian assumes the well-known DBI form,

$$\mathcal{L}(X, \phi) = -f^{-1}(\phi)\sqrt{1 - f(\phi)X} + f^{-1}(\phi) - V(\phi). \tag{74}$$

The DBI model with constant flow parameters is extensively discussed in Ref. [28], and the reader is referred this paper for further details. For ϵ and s constant, the functions V and f are fully determined and are given by

$$\begin{aligned} V(\phi) &= 3M_P^2H^2(\phi)\left[1 - \left(\frac{2\epsilon}{3}\right)\frac{1}{1 + c_S(\phi)}\right], \\ f(\phi) &= \left(\frac{1}{2M_P^2\epsilon}\right)\frac{1 - c_S^2(\phi)}{H^2(\phi)c_S(\phi)}. \end{aligned} \tag{75}$$

The Hubble parameter and speed of sound are given by

$$H(\phi) = H_0\left(\frac{\phi}{\phi_0}\right)^{-2\epsilon/s}, \tag{76}$$

and

$$c_S(\phi) = \left(\frac{\phi}{\phi_0}\right)^2. \tag{77}$$

DBI Lagrangians allow for either inflationary or tachyacoustic evolution [18], depending on the values of ϵ and s. Note that for $c_S > 1$, the function f is negative, which has consequences for embedding such a model in string theory, which we discuss in Sec. VII.

In this section, we have explicitly constructed Lagrangians, including fully determined potentials, for which the flow parameters are constant and the background evolution can be solved exactly. For suitable choices of the flow parameters, the evolution is tachyacoustic, i.e. with a growing comoving Hubble horizon and a shrinking comoving acoustic horizon. In the next section, we discuss the generation of curvature perturbations at the acoustic horizon and show that such perturbations are nearly scale-invariant, consistent with observation.

VI. COSMOLOGICAL PERTURBATIONS FOR CONSTANT FLOW PARAMETERS

We can deal with cosmological perturbations in this general k-essence model with constant flow parameters in the same way as performed in [28]. Following the approach of Garriga and Mukhanov [24] we start with the perturbed Einstein equations,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\delta\phi}{\dot{\phi}}\right) &= \Phi + \frac{2M_P^2c_S^2}{a^2(\rho + p)}\nabla^2\Phi, \\ \frac{d}{dt}(a\Phi) &= \frac{a(\rho + p)}{2M_P^2}\left(\frac{\delta\phi}{\dot{\phi}}\right), \end{aligned} \tag{78}$$

where Φ is the Bardeen potential and $\delta\phi$ is the perturbation of the field ϕ. Equations (78) can be cast into a more convenient form by changing the perturbations Φ and $\delta\phi$ to the new variables ζ and ξ defined by

$$\xi = \frac{2M_P^2\Phi a}{H}, \qquad \zeta = H\frac{\delta\phi}{\dot{\phi}} + \Phi, \tag{79}$$

so that the perturbed Einstein equations (78) become

$$\dot{\xi} = \frac{a(\rho + p)}{H^2}\zeta, \qquad \dot{\zeta} = \frac{c_S^2H^2}{a^3(\rho + p)}\nabla^2\xi. \tag{80}$$

As usual, we introduce a new variable z and the gauge-invariant Mukhanov potential u as

$$z = \frac{a(\rho + p)^{1/2}}{c_SH}, \qquad u = z\zeta; \tag{81}$$

then, from (80) we derive the mode equation for $u(\tau) \propto u_k(\tau)\exp(i\mathbf{k}\cdot\mathbf{x})$, given by

$$u_k'' - \left[(c_Sk)^2 + \frac{z''}{z}\right]u_k = 0, \tag{82}$$

where a prime denotes a derivative with respect to conformal time, $ds^2 = a^2(\tau)(d\tau^2 - d\mathbf{x}^2)$. It is easy to show that the variable z, defined by (81) can be cast into the following form,

$$z = -\frac{aM_P\sqrt{2\epsilon}}{c_S}; \tag{83}$$

then, using

$$\frac{d}{d\tau} = -aH\frac{d}{dN}, \tag{84}$$

we can evaluate the ratio z''/z in (82) in terms of the flow parameters (29)–(33); the result is

$$\frac{z''}{z} = a^2H^2\bar{F}(\epsilon, \tilde{\eta}, s, \tilde{s}, {}^2\lambda, {}^1\alpha, {}^1\beta), \tag{85}$$

where

$$\begin{aligned} \bar{F} \equiv{} & 2 + 2\epsilon - 3\tilde{\eta} - 3s + \frac{3}{2}\tilde{s} + 2\epsilon^2 + \frac{5}{4}\tilde{s}^2 - 2s\tilde{s} \\ & + \tilde{\eta}^2 + 2\epsilon(\tilde{s} - s) + 3\tilde{\eta}s - \frac{5}{2}\tilde{\eta}\tilde{s} - 4\tilde{\eta}\epsilon \\ & + {}^2\lambda - \frac{1}{2}\epsilon({}^1\alpha) + \epsilon({}^1\beta). \end{aligned} \tag{86}$$

Next, it is convenient to change the conformal time, τ, to the ratio of wave number to the sound horizon,

$$y \equiv \frac{c_S k}{aH}; \tag{87}$$

then, conformal time derivatives switch to

$$\frac{d}{d\tau} = -aH(1-\epsilon-s)y\frac{d}{dy}, \tag{88}$$

and

$$\frac{d^2}{d\tau^2} = a^2H^2\left[(1-\epsilon-s)^2y^2\frac{d^2}{dy^2} + \bar{G}(\epsilon,\tilde{\eta},s,\tilde{s},{}^1\alpha)y\frac{d}{dy}\right], \tag{89}$$

where

$$\bar{G} \equiv -s + \epsilon(2s+\tilde{s}) + s(2s+\tilde{s}) + 2\epsilon^2 - 2\epsilon\tilde{\eta} - s\tilde{\eta} - \epsilon({}^1\alpha). \tag{90}$$

It is important to stress that the functions F and G derived above hold in general; they reduce to the well-known expressions in the DBI limit [28], which, in this case, $s = \tilde{s}$ and ${}^1\alpha = {}^1\beta = \rho$. Substituting (85) and (89) into the mode equation (82), we find

$$(1-\epsilon-s)^2y^2\frac{d^2u_k}{dy^2} + \bar{G}y\frac{du_k}{dy} + [y^2 - \bar{F}]u_k = 0, \tag{91}$$

which is an exact equation, without any assumption of slow roll.

In the case where the flow parameters are constant, we can use the differential equations (32) to reduce the number of independent parameters. We have

$$\tilde{\eta} = \frac{1}{2}(2\epsilon+\tilde{s}), \qquad {}^2\lambda = \frac{1}{2}(2\epsilon+\tilde{s})(\epsilon+\tilde{s}),$$
$${}^1\alpha = \frac{s}{2\epsilon}(2s+\tilde{s}), \qquad {}^1\beta = \frac{3\tilde{s}^2}{2\epsilon}; \tag{92}$$

then, substituting these values into expressions (86) and (90), we find, respectively,

$$\bar{F} = 2 - \epsilon - 3s + \frac{9}{4}\tilde{s}^2 - \frac{3}{4}s\tilde{s} + \epsilon s - \frac{1}{2}s^2, \tag{93}$$

$$\bar{G} = s(-1+\epsilon+s). \tag{94}$$

It is important to notice that $\bar{F}$ is different from the corresponding expression found in the DBI case [28], since the gauge-dependent $\tilde{s}$ comes into play. However, $\bar{G}$ is identical to its DBI analog, and it is expected since basically it comes from the change of variables $\tau \rightarrow y$, which depends solely on the parameters c_S and H, and not on $\mathcal{L}_X$. For constant flow parameters we can solve Eq. (91) exactly, and the solutions are given by

$$u_k(y) = y^{((1-\epsilon)/2(1-\epsilon-s))}\left[c_1H_\nu^{(1)}\left(\frac{y}{1-\epsilon-s}\right) + c_2H_\nu^{(2)}\left(\frac{y}{1-\epsilon-s}\right)\right], \tag{95}$$

where c_1 and c_2 are constants, and $H_\nu^{(1)}$, $H_\nu^{(2)}$ are Hankel functions of first and second kind, respectively. The order ν of the Hankel function is given by

$$\nu^2 = \frac{9 - 6\epsilon - 12s + 9\tilde{s}^2 - 3s\tilde{s} + 4\epsilon s - 2s^2 + \epsilon^2}{4(1-\epsilon-s)^2}; \tag{96}$$

next, using (35), (84), and (88) we find that

$$c_S \propto y^{s/(\epsilon+s-1)}; \tag{97}$$

then, imposing the Bunch-Davies vacuum $c_2 = 0$, and normalizing the mode amplitudes by means of the canonical quantization condition

$$u_k^*\frac{du_k}{dy} - u_k\frac{du_k^*}{dy} = \frac{i}{c_sk(1-\epsilon-s)}, \tag{98}$$

we find

$$u_k(y) = \frac{1}{2}\sqrt{\frac{\pi}{c_sk}}\sqrt{\frac{y}{1-\epsilon-s}}H_\nu\left(\frac{y}{1-\epsilon-s}\right), \tag{99}$$

which differs from the DBI case only in the order of the Hankel function (96). In the small wavelength limit $y \rightarrow \infty$ the early-time behavior of u_k will be identical to DBI [28] for constant flow parameters

$$u_k = \frac{1}{\sqrt{2c_sk}}e^{iy/(1-\epsilon-s)}, \tag{100}$$

whereas in the late-time behavior $y \rightarrow 0$ the mode function behaves as

$$|u_k(y)| \rightarrow 2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}(1-\epsilon-s)^{\nu-1/2}\frac{y^{1/2-\nu}}{\sqrt{2c_sk}}. \tag{101}$$

From (101) we can derive the expression for the scalar spectral index n_s. Using the definition of the power spectrum of curvature perturbations

$$P_{\mathcal{R}}(k) = \frac{k^3}{2\pi^2}\left|\frac{u_k}{z}\right|^2, \tag{102}$$

and substituting expressions (83) and (101) into (102), we find

$$P_{\mathcal{R}} = \frac{|f(\nu)|^2}{8\pi^2M_P^2}\frac{H^2}{c_S\epsilon} \tag{103}$$

at horizon crossing, where $f(\nu)$ is a constant given by

$$f(\nu) = 2^{\nu-3/2}\frac{\Gamma(\nu)}{\Gamma(3/2)}(1-\epsilon-s)^{\nu-1/2}; \tag{104}$$

then, from the definition of the scalar spectral index

$$n_s - 1 \equiv \frac{d \ln P_{\mathcal{R}}}{d \ln k}, \tag{105}$$

and using

$$\frac{d}{d \ln k} = -\left(\frac{1}{1 - \epsilon - s}\right)\frac{d}{dN}, \tag{106}$$

we see that the spectral index n_s assumes the form

$$n_s = 1 - \frac{2\epsilon + s}{1 - \epsilon - s}, \tag{107}$$

which does not depend on the gauge-dependent parameter $\tilde{s}$, and is identical to its DBI analog. This is expected since the power spectrum evaluated at the horizon crossing, Eq. (103), depends solely on H and c_S, whose derivatives with respect to N are related to the gauge-invariant flow parameters ϵ and s. The scale-invariant limit is $s = -2\epsilon$.

VII. CONCLUSIONS

In this paper we have demonstrated that accelerated expansion or a collapsing universe are not the only ways to dynamically generate a scale-invariant spectrum of superhorizon curvature perturbations. There is a third way: a superluminal acoustic cone which is shrinking in comoving coordinates. Curvature perturbations generated at the acoustic horizon are familiar from inflationary scenarios based on noncanonical Lagrangians such as k-inflation and DBI inflation. Such noncanonical Lagrangians arise naturally in string theory. However, in these scenarios, *both* the Hubble horizon and the acoustic horizon are shrinking in comoving units, and the acoustic horizon is typically smaller than the Hubble horizon, i.e. $c_S < 1$. It is natural to ask whether tachyacoustic models have a similar, natural stringy embedding, especially since the DBI action (5) naturally admits tachyacoustic solutions. Such an embedding is nontrivial, however, since the frequently considered case of a $3 + 1$ dimension d-brane evolving in a higher-dimensional throat is ill defined in the $c_S > 1$ limit. To see this, consider the full ten-dimensional metric of throat plus brane [30],

$$ds_{10}^2 = h^2(r) ds_4^2 + h^{-2}(r)(dr^2 + r^2 ds_{X_5}^2). \tag{108}$$

The field ϕ is simply related to the coordinate in the throat r as $\phi = \sqrt{T_3} r$, where the brane tension T_3 depends on the string scale m_s and the string coupling g_s as [31]

$$T_3 = \frac{m_s^4}{(2\pi)^3 g_s}. \tag{109}$$

The Lagrangian for the field ϕ can be shown to be of the DBI form (5), where the inverse brane tension $f(\phi)$ is given in terms of the warp factor $h(\phi)$ by

$$f(\phi) = \frac{1}{T_3 h^4(\phi)}. \tag{110}$$

The problem is immediately evident: superluminal propagation $c_S > 1$ requires $f < 0$, so that the factor $h^2(\phi)$ appearing in the metric (108) is imaginary, and the metric is ill defined. Therefore, although the DBI action itself admits tachyacoustic solutions, this limit does not correspond to a well-defined string solution. It is not clear whether or not string manifolds exist which self-consistently admit solutions with $c_S > 1$.

We calculate the scalar spectral index of perturbations for tachyacoustic solutions, and find

$$n_s = 1 - \frac{2\epsilon + s}{1 - \epsilon - s}. \tag{111}$$

Unlike inflationary models, radiation-dominated tachyacoustic models do not require a period of explosive entropy production to transition to a "hot" big bang cosmology. The early Universe must be scalar-field dominated, but the temperature of the Universe is not driven exponentially to zero, since the scalar has a radiation equation of state at all times, and entropy density is conserved (for any radiation component with density ρ_γ, the ratio $\rho_\phi/\rho_\gamma = \text{const}$). The scalar field ϕ must eventually decay to standard model degrees of freedom, but as long as this happens before primordial nucleosynthesis, the model will match observations. A slow or late decay of ϕ into other degrees of freedom would also suppress the production of unwanted relics such as monopoles or gravitinos. For radiation-dominated tachyacoustic expansion with $\epsilon = 2$, the spectral index is

$$n = 1 + \frac{4 + s}{1 + s}, \tag{112}$$

where we have $s < -3$ for a shrinking comoving acoustic cone. For $s < -4$, the spectral index is blue, $n > 1$, which is ruled out by observation. The WMAP 2σ limit $n = 0.96 \pm 0.026$ [5] corresponds to $s = [-3.814, -3.959]$. Since the Hubble horizon is growing in comoving units, no gravitational wave modes are produced.

Tachyacoustic models are not a fully convincing alternative to inflation, since they solve only the horizon problem and not the flatness problem, and inflation solves both at once. However, inflation has initial conditions problems of its own, in particular, the fact that the initial inflationary "patch" must be larger than a horizon size for inflation to start [32]. Furthermore it has been shown that inflationary spacetimes are in general geodesically past-incomplete [33]. The initial conditions for tachyacoustic cosmology are quite different than those for inflation due to the presence of a true "big bang" singularity at zero time. However, in this limit, the sound speed is *infinite* and the tachyacoustic solution approaches an instanton. To see this, examine the form of the DBI field Lagrangian (5) near the $\tau = 0$ boundary of a tachyacoustic spacetime. From Eq. (12), the $c_S \to \infty$ limit corresponds to $\phi \to \infty$ and $f(\phi)\dot{\phi}^2 \to -\infty$, so that

$$\mathcal{L} \to \frac{\dot{\phi}}{\sqrt{|f|}} - V(\phi), \tag{113}$$

where

$$\dot{\phi} \equiv \sqrt{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi}. \tag{114}$$

From Eq. (75), the asymptotic behavior of $V(\phi)$ and $f(\phi)$ are

$$\begin{aligned} V(\phi) &\to 3M_P^2H^2 \propto \phi^{-4\epsilon/s}, \\ f(\phi) &\to -\frac{1}{2M_P^2\epsilon}\frac{c_S}{H} \propto \phi^{2(1+2\epsilon/s)}. \end{aligned} \tag{115}$$

The scale-invariant limit $s = -2\epsilon$ is especially interesting, since

$$\frac{1}{\sqrt{|f|}} \to \mu^2 = \text{const}, \tag{116}$$

and the Lagrangian takes the form

$$\mathcal{L} \to \mu^2\dot{\phi} - V(\phi), \tag{117}$$

where $V(\phi) \propto \phi^2$. This can be identified as exactly the cuscuton Lagrangian, suggested by Afshordi, *et al.* as a candidate for dark energy [29,34,35]. Similarly, the $n = 0$ solution considered in Sec. V approaches a cuscuton on the initial boundary surface. The cuscuton is a nondynamical, instantonlike solution with infinite speed of sound. Consider the action for the Lagrangian (117),

$$\begin{aligned} S_\phi &= \int d^4x\sqrt{-g}[\mu^2\dot{\phi} - V(\phi)] \\ &= \mu^2\int d\phi\Sigma(\phi) - \int d^4x\sqrt{-g}V(\phi), \end{aligned} \tag{118}$$

where $\Sigma(\phi)$ is the volume of a constant-ϕ hypersurface in the spacetime. The classical solutions to the cuscuton action are constant mean curvature hypersurfaces, analogous to soap bubbles [29]. It is interesting to speculate that this property of the cuscuton action may provide a self-consistent cosmological boundary condition, or (even more speculatively) be useful as a solution to the cosmological flatness problem. A full analysis, however, would require inclusion of the gravitational action and solution in a Wheeler-DeWitt framework, or perhaps an embedding of the model in string theory or an alternate gravity theory such as Horava-Lifshitz [36,37]. This is the subject of future work.

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APPENDIX: STABLE CAUSALITY

Since tachyacoustic cosmology deals with superluminal propagation of perturbations, it is important to address the issue of *causality* in this model. Babichev *et al.* [26] have discussed the conditions that must be fulfilled by a general k-essence model with superluminal propagation in order to avoid causal paradoxes (i.e., the presence of *closed causal curves*—CCC). In this appendix we outline the main ideas of this work and apply to our tachyacoustic model.

To begin with let us introduce some key definitions [38]. Let $g_{\mu\nu}$ be a metric with Lorentzian signature defined on a given manifold $\mathcal{M}$. Given a point $p \in \mathcal{M}$, let t^μ be a timelike vector at p; then, from this timelike vector we construct a second metric, $\tilde{g}_{\mu\nu}$, related to the background metric $g_{\mu\nu}$ by

$$\tilde{g}_{\mu\nu} \equiv g_{\mu\nu} - t_\mu t_\nu. \tag{A1}$$

The spacetime $(\mathcal{M}, g_{\mu\nu})$ is defined to be *stably causal* if there is a continuous timelike vector field t^μ such that the spacetime $(\mathcal{M}, \tilde{g}_{\mu\nu})$ possesses no closed timelike curves. The following theorem (8.2.2. in [38]) establishes the necessary and sufficient conditions for a spacetime to be stably causal:

A spacetime $(\mathcal{M}, g_{\mu\nu})$ *is stably causal if and only if there exists a differentiable function* f *on* $\mathcal{M}$ *such that* $\nabla^\mu f$ *is a past directed timelike vector field.*

We can apply this theorem to k-essence models as follows [26]. First, we must find the analog of the induced metric (A1) for the case of k-essence models, which can be obtained by means of the equation of motion for a scalar field described by a Lagrangian $\mathcal{L}(X, \phi)$,

$$\tilde{G}^{\mu\nu}\nabla_\mu\nabla_\nu\phi + 2X\mathcal{L}_{X\phi} - \mathcal{L}_\phi = 0, \tag{A2}$$

where $\tilde{G}^{\mu\nu}$, called "effective" or "acoustic" metric is given by

$$\tilde{G}^{\mu\nu}(\phi, \nabla\phi) = \mathcal{L}_X g^{\mu\nu} + \mathcal{L}_{XX}\nabla^\mu\phi\nabla^\nu\phi. \tag{A3}$$

It is convenient to use the metric [26]

$$G^{\mu\nu} \equiv \frac{c_S}{\mathcal{L}_X^2}\tilde{G}^{\mu\nu} \tag{A4}$$

which is conformally equivalent to $\tilde{G}^{\mu\nu}$, and hence, defines the same causal structure. The inverse metric $G^{-1}_{\mu\nu}$ is given by

$$G^{-1}_{\mu\nu} \equiv \frac{\mathcal{L}_X}{c_S}\left[g_{\mu\nu} - c_S^2\frac{\mathcal{L}_{XX}}{\mathcal{L}_X}\nabla_\mu\phi\nabla_\nu\phi\right]; \tag{A5}$$

notice that it has the same form of (A1), since $\nabla^\mu\phi$ is timelike. Using this definition, we can now apply the theorem stated above and check the stable causality of k-essence models. Let t be time coordinate with respect to the background metric (which is everywhere future directed), which we take to be FRW. Since $g^{\mu\nu}\nabla_\mu t\nabla_\nu t = 1$, we have, using (A3) and (A4),

$$G^{\mu\nu}\nabla_\mu t\nabla_\nu t = \frac{c_S}{\mathcal{L}_X}\left[1 + \frac{\mathcal{L}_{XX}}{\mathcal{L}_X}\dot{\phi}^2\right]; \tag{A6}$$

then, since for a *homogeneous* scalar field holds $\dot{\phi}^2 = 2X$, we have, from (18) and (A6) that

$$G^{\mu\nu}\nabla_\mu t\nabla_\nu t = \frac{1}{c_S\mathcal{L}_X} > 0, \tag{A7}$$

provided the null energy condition (NEC) is satisfied, that is, $\mathcal{L}_X > 0$. Therefore, t plays a role of global time for *both* spacetimes $(\mathcal{M}, g_{\mu\nu})$ and $(\mathcal{M}, G^{-1}_{\mu\nu})$, and then the conditions of the theorem are fulfilled. Then, there is *no* CCC in superluminal k-essence models built from homogeneous scalar fields on a FRW background. Since this is exactly the case of the models introduced in this paper, we conclude that there are no causal paradoxes in tachyacoustic cosmology.

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