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CONTROLLING SELF-ORGANIZED CRITICALITY IN COMPLEX NETWORKS

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Introduction The phenomenon of self-organized criticality (SOC) on complex networks has recently been studied in order to understand the failures that take place in real networks such as power transportation and internet networks [1]. The main idea is that, due to the strong relation among the neighbors, a small catastrophe that occurs in a node or a small collection of nodes may be spread to the whole network. Parallel to this, there is a growing literature that deals with the issue of robustness and the reduction of overload failure cascades caused by attacks or overload in complex networks [2]. The remedy often used in order to avoid the propagation of these cascades is the intentional removal of some special nodes characterized by their degree or by a given centrality measure.

Here, we propose a control scheme to reduce the probability of large avalanches in self-organized critical systems that happen in complex networks. The control scheme is based on the triggering of avalanches on a percentage of the highest degree nodes.

The issue of controlling SOC in regular lattices, where mass is added and removed from a system, has been recently discussed in sandpile models [3]. In that paper, we have shown that an external control action, which amounts to triggering avalanches in sites that are near to be come critical, was able to reduce the probability of very large events, so that mass dissipation occurs most locally. Due to the homogeneity of the lattice where traditional SOC phenomena have been investigated, one difficult present in [3] is that, in order to make the decision whether an avalanche should or not be exogenously triggered, one had to simulate a replica model of the region of the system to be controlled. Here, differently from [3], the control scheme does not depend on the replica model and, therefore, is less costly than the one presented in [3]. Furthermore, while in [3] we were interested in controlling the size of avalanches in only a region of the system, in this paper we are interested in controlling the size of the avalanches in the whole network.

Controlled BTW model in complex networks The Bak-Tang-Wiesenfeld (BTW) sandpile model [4] has been recently studied in scale-free networks [5] by [1]. Here, we follow closely this previously developed approach. Consider a network with n nodes. Let $k(i)$ be the degree of node $i \in \{1, \cdots, n\}$ and $\mathcal{N}(i)$ be the set of neighbors of node i. Assume also that each node $i \in \{1, \cdots, n\}$ stores a certain amount z_i of mass units. The dynamics of the BTW model in complex networks may be described by the following two rules: (a) Addition rule: at each time step, a mass unit is added to a randomly selected node $i \in \{1, \cdots, n\}$, so that $z_i \to z_i + 1$. (b) Toppling rule: if $z_i \geq z_{ic} = k(i)$, then $z_i \to z_i - k(i)$, $z_j \to z_j + 1, \forall j \in \mathcal{N}(i)$.

In order to control the BTW model, we propose here the so-called Highest Degree Nodes Control Based (H-control), which assumes that we have global knowledge of the network. The idea is to choose the percentage of highest degree nodes p_H of the network that will be controlled and to build a set $\mathcal{S}_H$ with these nodes. To be controlled here means that if $z(i) = z_c(i) - 1$ of a node $i \in \mathcal{S}_H$ in a given time, the control system triggers an explosion on this node. This means that a real controlled[1] avalanche is triggered by emptying the node i and the mass available in this site goes randomly to some of the neighbors belonging to $\mathcal{N}(i)$. When it is the case, first we trigger an avalanche in the highest degree node, then we trigger in the second highest degree node and so on. We compare this control scheme with the so-called Random Selected Nodes Control Based (R-control), which assumes no information about the network, but it intervenes with the same frequency of the H-control. In each instant of time, it selects randomly the nodes to be controlled and triggers an explosion on this node, if $z(i) = z_c(i) - 1$. Both control schemes assume that we keep the same mass in the system. Unfortunately, we cannot prove optimality of any of these strategies, since the mathematical model associated to this system is very complicated being a large set of non-linear coupled difference equations. In fact, in order to reach optimality, one should deal with partial removal of the demand from the

[1]We call here the avalanches that occur due to the intervention of the control scheme as *controlled avalanches* in order to differentiate them from the *uncontrolled avalanches* that happen due to the deposition of mass in SOC dynamics.

nodes that are likely to become critical and consider all the possible order of triggering the avalanches. Partial removal may work worse than both strategies. Although partial removal may avoid avalanches created by the control scheme, it can allow the system to accumulate energy that in the future can cause larger avalanches. Therefore, we only intend to show that it is possible to reduce the size of avalanches on complex networks.

Results We have a applied this control scheme to BTW model on scale-free networks built based on the algorithm provided by [5] that can be described as follows: Start with n nodes $i \in \{1, \cdots, n\}$ and assign to each of them a weight equal to $w_i = i^{-\alpha}$, where $\alpha \in [0, 1]$ is related to the degree exponent according to $\gamma = 1+1/\alpha$. Then select two different nodes $i, j \in \{1, \cdots, n\}$ with probability equal to the normalized weights $w_i / \sum_{k=1}^{n} w_k$ and $w_j / \sum_{k=1}^{n} w_k$, respectively, and connect them if they are not already connected. The exponent of the avalanche size distribution in these scale-free networks was determined to be $\tau = \gamma/(\gamma - 1)$ [1].

Fig.1 compares the the probability distribution function (PDF) of avalanche sizes $p(s)$ of the uncontrolled system (solid symbols) with that of the system controlled by the H-control (hollow symbols). While the data of the uncontrolled system include only the uncontrolled avalanches, those of the controlled system include controlled and uncontrolled avalanches. The PDF of the degrees of the nodes of the GKK networks presented in this figure is a power-law with theoretical exponent $\gamma = 3.0$, since we used the value of $\alpha = 0.5$ to build them. For both networks with 10^4 and 10^5 nodes we have numerically obtained the exponent $\hat{\gamma} = 3.07$. The straight line in Fig. 1 is the best fit to the data for the systems with size $n = 10^4$ and 10^5 in the interval $s \in [10^{0.3}, 10^{3.3}]$. Based on this data, the exponent of the avalanche size distribution in these GKK networks was determined to be $\hat{\tau} = 1.74$ which is roughly the value of the empirically one determined in [1]. Fig.1 also shows that the H-control is able to strongly reduce the probability of large events. Besides, this figure shows that when p_H increases, the control scheme is more efficient in the reduction of large size avalanches. Finally, we also compare the efficiency of H-control scheme with R-control scheme. We can see that although both are efficient to reduce the probability of large avalanches, the former is much more efficient. The decrease in the probability of large avalanches results from the fact that, since only saturated sites are exploded by random process, some of them sites are correctly chosen. Furthermore, since the most connected nodes have by definition more neighbors, even if the explosion are wrongly selected, these explosions are likely to have some effect in the most connected nodes.

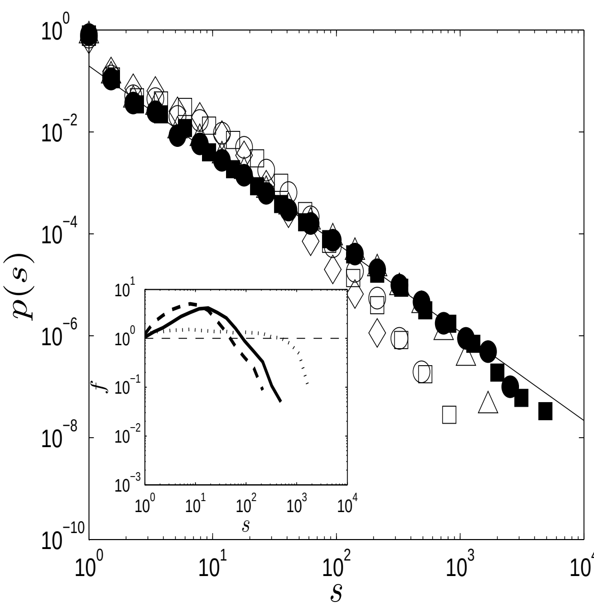

Figure 1 – Probability distribution of avalanche sizes $p(s)$ in the GKK network with mean average degree 4. Points were obtained by logarithmic size bins over the whole range of s. Solid and hollow symbols denote uncontrolled and controlled system, respectively. Uncontrolled system sizes $n = 10^5$ (squares) and 10^4 (circles). Symbol types indicate the following values of $(n, \text{control scheme (H or R)}, p_H, N_T)$ for the controlled systems: squares (10^5, H, 5%, 0.27), circles (10^4, H, 5%, 0.25), diamonds (10^4, H, 10%, 0.52) and triangles (10^4, R, – , 0.25). In the inset, curves for the ratio f between total number of avalanches in the controlled and uncontrolled simulations. The number of time steps are equal for both simulations. Line types are as follows. Scale-free network: solid $(10^4, \mathrm{H}, 5\%, 0.25)$, dashes $(10^4, \mathrm{H}, 10\%, 0.52)$ and dots $(10^4, \mathrm{R}, 5\%, 0.25)$. The curve for ($10^5$, H, 5%, 0.27) was not shown since it is difficulty to differentiate this curve from the solid one.

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