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Characterizing the common behavior close to stickiness in Hamiltonian systems

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1. PLAN AND MOTIVATION

In this work we study and compare two Hamiltonian maps (global and local couplings) with the symplectic map of Ref. [1] where an unidirectional coupling was used. Basically N coupled standard maps are used. The main subject in work [1] was the investigation of how stickiness effects are distributed in different dimensions and the existence of a kind "common behavior"for the finite time Lyapunov exponents (FTLE) distribution at small nonlinearities. As the nonlinearity increases, it was clearly identified the transition from quasi-regular to totally chaotic motion which occurs simultaneously in all unstable directions. The present work investigates and explains the common behavior for other coupling models.

2. QUANTITATIVE CHARACTERIZATION OF THE DISTRIBUTIONS

The presence of stickiness in the mixed phase space of conservative systems is difficult to detect and to characterize, in particular for high dimensional phase spaces. In the quasi-regular regime the sticky motion influences the distribution of the finite time Lyapunov exponents qualitatively (we denote by $\lambda_n^{(k)}$ the FTLE with k being the number of positive Lyapunov exponents). The mentioned influence was quantified in the work [1] with four variables: the variance (and the higher cummulants, skewness and kurtosis) and the normalized number of occurencies of the most probable FTLE. We study systematically standard maps (symplectic systems) beginning with the uncoupled two-dimensional case up to coupled maps of dimension $d = 20$. Using the four variables we find that the effect of the sticky motion on the distributions of Lyapunov exponents is equal in different unstable directions above a threshold K_d of the nonlinearity parameter K for the high dimensional cases $d = 10, 20$. Clearly, anomalies in the FTLE distribution reflect sticky motion. Here we analyze the higher cummulants of the FTLE defined by:

a.) The *variance* defined by $\tilde{\sigma} = \left\langle (\lambda_n - \langle \lambda_n \rangle)^2 \right\rangle$, which is the second cummulant of the FTLE distribution. It should increase for sticky motion. We will analyze the properties of the relative variance $\sigma = \tilde{\sigma} / \langle \lambda_n \rangle^2$ which is more appropriate for higher-dimensional systems. In such cases it is possible to detect small differences relative to each unstable direction.

b.) The *skewness* is defined by $\kappa_3 = \dfrac{\left\langle (\lambda_n - \langle \lambda_n \rangle)^3 \right\rangle}{\tilde{\sigma}^{3/2}}$. It is the third cummulant of the FTLE distribution and detects the asymmetry of the distribution around its mean value. For $\kappa_3 = 0$ we have the regular distribution. Since sticky motion usually reduces the FTLE, the asymmetry of the distribution leads to $\kappa_3 < 0$.

c.) The *kurtosis* is defined by $\kappa_4 = \dfrac{\left\langle (\lambda_n - \langle \lambda_n \rangle)^4 \right\rangle}{\tilde{\sigma}^2} - 3$, and detects the shape of the distribution. where $\kappa_4 > 0$ indicates that the distribution is flatter than the regular distribution ($\kappa_4 = 0$) while $\kappa_4 < 0$ reveals a sharper distribution than the normal one.

These three quantities will be used to characterize the degree of sticky motion in the transition from low- to higher-dimensional systems. Altogether they should give all relevant informations which can be extracted from the distributions.

3. COUPLED STANDARD MAPS

Coupled standard maps are convenient systems to investigate the sticky motion in high-dimensional phase spaces systematically. The equations for N coupled standard maps read

$$q_{n+1}^{(i)} = q_n^{(i)} + p_{n+1}^{(i)}, \qquad p_{n+1}^{(i)} = p_n^{(i)} + f(q_n), \qquad (1)$$

with $f(q_n)$ being the global coupling (GC) or local coupling (LC) defined respectively according to

$$f(q_n) = \frac{K}{\sqrt{N-1}} \sum_{j=1, j\neq i}^{N} \sin(q_n^{(j)} - q_n^{(i)}),$$

$$f(q_n) = \frac{K}{\sqrt{2}} [\sin(q_n^{(i+1)} - q_n^{(i)}) - \sin(q_n^{(i)} - q_n^{(i-1)})],$$

where $i = 1, \ldots, N$ and $q \in [-\pi, \pi]$ [2]. Besides the symplectic area, the system given by Eqs. (1) also preserves the total momentum $\sum_{i=1}^{N} p_{n+1}^{(i)} = \sum_{i=1}^{N} p_n^{(i)}$. Therefore we always have two Lyapunovs equal zero. The phase space trajectories lie on a $(2N-2)$-dimensional hyper-space.

4. RESULTS

For the case $N = 3$ both couplings (GC and LC) are identical. This case has six FTLEs λ_n^k ($k = 1 \rightarrow 6$): two positive, two zero and two negative. The analysis will be done for the distributions of the two positive FTLEs which are labeled such that $\lambda_n^1 > \lambda_n^2$. Figure 1 shows the cummulants:

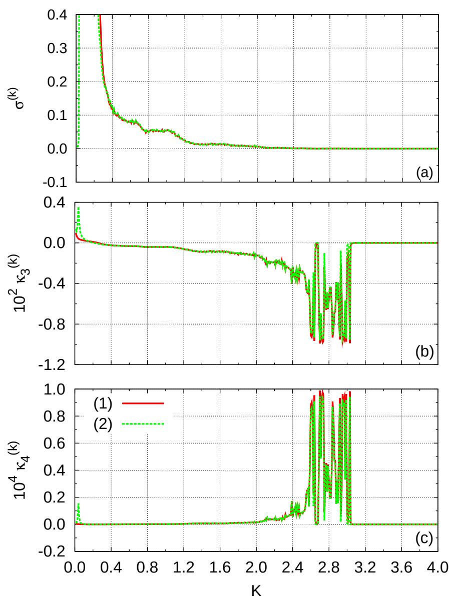

Figura 1 – Comparation of (a) $\sigma^{(k)}(K)$, (c) $\kappa_3^{(k)}(K)$ and (d) $\kappa_4^{(k)}(K)$ in the interval $K = (0.01, 4.0)$ and $N = 3$ for $k = 1, 2$.

(a) $\sigma^{(k)}$, (b) $\kappa_3^{(k)}$ and (c) $\kappa_4^{(k)}$ as function of K for $\lambda_n^{(k=1)}$ (red-full line) and $\lambda_n^{(k=2)}$ (green-dashed line). By comparing these plots we see that both distributions detect the sticky motion for $0.25 \leq K \leq 3.0$, where $\kappa_3^{(k)} < 0$ and $\kappa_4^{(k)} > 0$.

Figure 2 shows the behavior for three different values of K: for $K = 3.50$ no common motion was detected; in the case of $K = 0.29$ and 0.28 we can see some regions where the local FTLEs collapses to zero because the existence of stickiness effect.

Figure 3 shows the momenta p_n^k ($k = 1, 2, 3$). The black

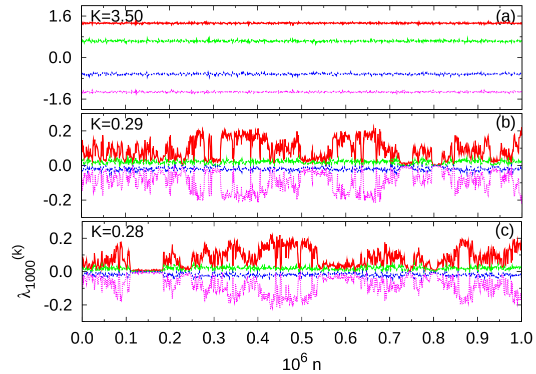

Figura 2 – Time evolution of local FTLE $\lambda_{1000}^{(k)}$ ($k = 1, 2, 5, 6$) for one exemplary trajectory with $N = 3$ for three different values of K.

boxes display two sticky regions for $K = 0.29$ and 0.28, where the momentum of each site becomes constant.

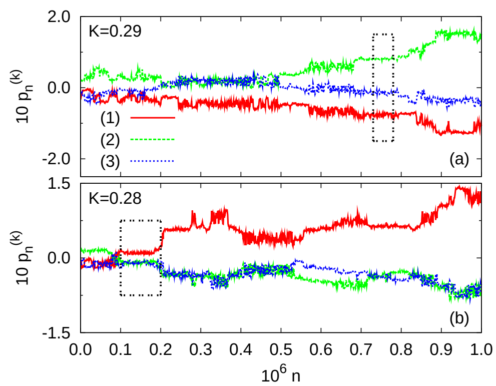

Figura 3 – Time evolution of $p^{(k)}$ ($k = 1, 2, 3$) for the same trajectory from Fig. 2. Black boxes show the sticky regions.

5. CONCLUSIONS AND ACKNOWLEDGMENTS

Extensive numerical simulations were also performed for $N = 5, 10$ (global and local couplings). We conclude saying that in all cases we found that the commonn behavior close to stickiness occurs for small nonlinearities and is always related to the conservation of the individual momentum of each site. Therefore the common behavior is a general feature in higher dimensional Hamiltonian systems.

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Referências

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