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GENERALIZING THE LOGISTIC MAP THROUGH THE q-PRODUCT

Robson Wilson Silva Pessoa[1] *and Ernesto Pinheiro Borges*[2]

[1]UFBA, Escola Politécnica, Salvador, Brazil, robsonpessoa@ufba.br
[2]UFBA, Escola Politécnica, Salvador, Brazil, ernesto@ufba.br

Abstract: We investigate a generalization of the logistic map as $x_{n+1} = 1 - ax_n \otimes_{q_{map}} x_n$ $(-1 \le x_n \le 1,\, 0 < a \le 2)$ where $\otimes_q$ stands for a generalization of the ordinary product [Borges, E.P. Physica A **340**, 95 (2004)]. The usual product is recovered in the limit $q \to 1$, and consequently the usual logistic map is recovered. The tent map is also a particular case at $q_{map} \to \infty$. The generalization of this (and others) algebraic operator has been widely used within nonextensive statistical mechanics context. We focus the analysis at the edge of chaos, particularly at the first critical point, that depends on the value of q_{map}. Bifurcation diagrams, sensitivity to initial conditions, fractal dimension and rate of entropy growth are evaluated at $a_c(q_{map})$. Connections with nonextensive statistical mechanics are explored.

keywords: logistic map, nonextensivity, edge of chaos

1. THE q-LOGISTIC MAP

Low-dimensional non-linear maps represent paradigmatic models in the analysis of dynamic systems. The discrete time evolution and the small number of relatively simple equations make their treatment easy, without losing the richness of the behavior, exhibiting order, chaos and a well defined transition between them.

Strongly chaotic systems are of special interest for statistical mechanics, once they feature well known characteristics: exponential sensitivity to the initial conditions, exponential relaxation to the equilibrium state, gaussian distributions.

In-between ordered systems (with negative Lyapunov exponent) and (strongly) chaotic systems (with positive Lyapunov exponent), there are those with zero maximal Lyapunov exponent. These systems are characterized by power-law sensitivity to initial conditions, instead of the exponential sensitivity, and thus are considered as weak chaotic systems. This change in the dynamics may lead to break of ergodicity, non exponential relaxation to equilibrium and non gaussian distributions. These behaviors are usually expected to be found in systems that are described by nonextensive statistical mechanics [1, 2]. Some low dimensional maps, *e.g.* the logistic map, also exhibit weak chaoticity at the edge of chaos, and hence the interest in studying them to better understand nonextensivity.

Power-law like sensitivity to initial conditions and power-law like relaxation to the attractor (more precisely a q-exponential law) have already been found in logistic-like maps [3, 4]. q-exponential function ($e_q^x \equiv [1 + (1 - q)x]^{1/(1-q)}$) appear within nonextensive statistical mechanics and it generalizes the usual exponential function (recovered as $q \to 1$). It is asymptotically a power-law. Sensitivity to initial conditions of the logistic map at the edge of chaos is identified to a q-exponential, with a specific value of the parameter q, denoted as q_{sen}. Relaxation of the logistic map to the attractor at the edge of chaos also follows a q-exponential behavior, with a different and specific value of the parameter q, denoted as q_{rel}. The relation between $q_{sen} \le 1$ and $q_{rel} \ge 1$ plays a central role in the foundations nonextensive statistical mechanics. For completely chaotic systems, where Boltzmann-Gibbs formalism applies, these values collapse to $q_{sen} = q_{rel} = 1$ (See [2] for details).

Nonextensive statistical mechanics has lead to developments in many neighboring areas, including generalized algebras [5, 6]. These works have introduced generalized algebraic operators, and here we are particularly interested in the q-product, defined as

$$x \otimes_q y \equiv \mathrm{sign}(x)\mathrm{sign}(y)\left[|x|^{1-q} + |y|^{1-q} - 1\right]_+^{\frac{1}{1-q}} \quad (1)$$

where $[A]_+ \equiv \max\{0, A\}$. If the limit $q \to 1$ is taken, the usual product ($x \otimes_1 y = xy$) is recovered.

Our work consists in generalizing the logistic map as

$$\begin{aligned} x_{n+1} &= 1 - ax_n \otimes_{q_{map}} x_n \\ &= 1 - a[2|x|_n^{1-q} - 1]_+^{\frac{1}{1-q}} \end{aligned} \quad (2)$$

$(-1 \le x_n \le 1,\, 0 < a \le 2)$. It is expected that values of q_{sen} and q_{rel} depend on q_{map}. We numerically pursue these dependencies, with the hope that these relations may shed some light on the multiplicity of q values and how they are related to each other.

The q-logistic map generalizes the usual logistic map for $q_{map} = 1$, and also the tent map for $q \to +\infty$. At the limit $q \to -\infty$, it becomes $x_{n+1} = 1 - a$. Figure 1 shows one iteration of the map.

Figure 2 shows bifurcation diagrams for different values of q_{map}. Windows of order inside chaos (where there is tan-

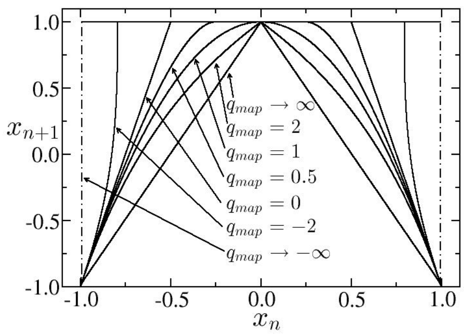

Figure 1 – x_{n+1} as a function of x_n for the q-logistic map. The usual parabolic behavior is recovered at $q_{map} = 1$. Tent map is found at $q \to \infty$. For $q_{map} < 1$, the cut-off condition in the definition of the q-product introduces a discontinuity in the map. For $q_{map} \to -\infty, \forall x_0 \in [-1, 1]$ leads to $x_1 = x_2 = \cdots = x_n = 1$.

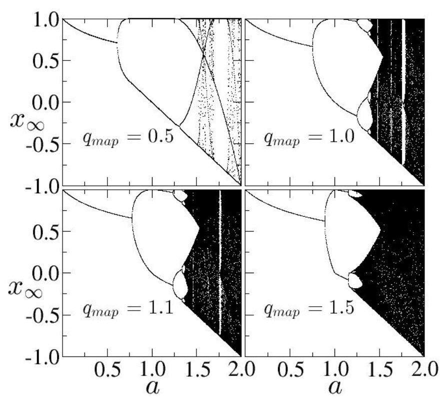

Figure 2 – Bifurcation diagram for different values of q_{map} (indicated). Windows of order inside chaos vanishes as $q_{map} \to 2$.

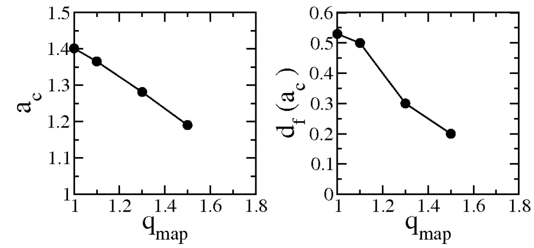

Figure 3 – First accumulation point and fractal dimension as a function of q_{map}. Lines are guide to the eyes.

gent bifurcation) gets narrower and completely disappears at $q_{map} = 2$. Complete chaos is preserved at $a = 2$, $\forall q_{map}$.

Figure 3 shows the parameter a for the first point of accumulation of bifurcations (a_c) and the fractal dimension at $a = a_c$ for some values of q_{map} (values found with box counting method). We see that fractal dimension decreases with increasing q_{map}.

Other properties, like sensitivity to initial conditions and relaxation to the attractor, are under evaluation.

References

[1] C. Tsallis; J. Stat. Phys. **52**, 479 (1988).

[2] C. Tsallis, Introduction to Nonextensive Statistical Mechanics, Springer, NY, 2009

[3] C. Tsallis, A.R. Plastino, W.-M. Zhen; Chaos, Solitions and Fractals **8**, 885 (1997).

[4] F.A.B.F. de Moura, U. Tirnakli, M.L. Lyra; Phys. Rev. E **62** (5), 6361 (2000).

[5] L. Nivanen, A. Le Méhauté , Q.A. Wang, Rep. Math. Phys. **52**, 437 (2003).

[6] E.P. Borges; Physica A **340**, 95 (2004).