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FERMI-ULAM MODEL: GENERIC DESCRIPTION FOR A DISSIPATIVE VERSION

Danila Fernandes Tavares de Sousa[1], *Raimundo Nogueira da Costa Filho*[2], *Edson Denis Leonel*[3]

[1]Universidade Federal do Ceará - UFC, Fortaleza, Brazil, dftavares@fisica.ufc.br
[2]Universidade Federal do Ceará - UFC, Fortaleza, Brazil, raimundo.costafilho@gmail.com
[3]Universidade Estadual Paulista - UNESP, Rio Claro, Brazil, edleonel@rc.unesp.br

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In this work we study the one-dimensional Fermi accelerator model in two versions, complete and simplified, in the presence of a small amount of dissipation, a viscous drag force, like a gas. The model consists of a classical particle of mass m, confined and colliding elastically between two rigid walls, one fixed and the other moving periodically in time (complete version)[1]. In the simplified version [2], the two walls are assumed to be fixed, but at the collision moment the particle suffers a transfer of energy and momentum with the wall, as if it was moving. This simplification allows us to reduce considerably the computational time spent on numerical simulations, because it is no more necessary to solve transcendental equations, as in the complete version, where one wall is movable. The results obtained in the two versions are very similar, keeping the nonlinearity of the problem. Thus, for most of our simulations, we will use the simplified version.

The problem is described in terms of a two dimensional mapping $T(v_n, t_n) = (v_{n+1}, t_{n+1})$. The particle suffers the effect of a frictional force whose magnitude is assumed to be of the type $F = -\eta v^{\gamma}$, where η is the coefficient of viscosity v the particle's velocity and γ a generic exponent that, in principle, can assume any integer value, except 1 and 2. The results for this problem considering $\gamma = 1$ and γ can be found in the literature. For a damping force proportional to the particle, $\gamma = 1$, Leonel and McClintock [3] showed an average linear decay of the particle's velocity with increasing the number of iterations. The Jacobian matrix sometimes shrinks the area in phase space and sometimes preserves area in the phase space. This fact is related to the Poincare's Theorem [4], which is a consequence of Liouville's Theorem that is not satisfied. The theorem says that, for a phase space bounded, almost all orbits return arbitrarily close to where they departed.

For the case where the drag force is assumed to be directly proportional to the square of the particle's velocity, $\gamma = 2$ [5, 6], it was observed an exponential decay of the particle's velocity with the increasing the number of iterations and their approach to a fixed attractive point is also exponential. It was shown that the system does not preserve area of phase space, in both complete and simplified versions and the well-known mixed structure of phase space for the case where there is no dissipation introduced in the system (invariant spanning curves, KAM islands and sea of chaos), it is totally destroyed, leading to a large number of attractive fixed points. The basin of attraction, which is the region access of evolution of an initial condition for the periodic orbits was constructed and the existence of many attractors in the system produced a basin of attraction structure very complex and extremely sensitive to initial conditions.

Thinking about the different results presented in the models for $\gamma = 1$ and $\gamma = 2$, that a small change in velocity from one model to another is able to generate, we decided to continue this discussion by proposing a model whose viscous drag force is proportional to a generic value of velocity. To construct the map, we assume that at time $t = t_n$ the particle is at position $x = 0$ with velocity $v = v_n > 0$. After solving Newton's law $-\eta v^{\gamma} = m dv/dt$, considering the effects of elastic collisions with both walls and defining dimensionless variables $V_n = v_n/(\omega l)$, $\varepsilon = \epsilon/(l)$, $\delta = \eta(\gamma-1)/\omega(\omega l)^{(1-\gamma)}$ and $\phi_n = \omega t_n$, we find the mapping given by

$$\mathrm{T}: \begin{cases} V_{n+1} = -(V_n{}^{\star} + \delta\phi_c)^{\frac{1}{(1-\gamma)}} - 2\epsilon\sin(\phi_{n+1}) \\ \phi_{n+1} = \phi_n + \Delta T_n \mod (2\pi) \end{cases}, \quad (1)$$

where expressions $V_n{}^{\star}$ and ΔT_n are given according to the type of collision that occurs, as follows:

(i) Multiple Collisions: the particle suffers more from a collision with the moving wall before leaving the collision zone, at $x \in [-\epsilon, \epsilon]$. In this case, $V_n{}^{\star} = V_n{}^{(1-\gamma)}$ and $\Delta T_n = \phi_c$. The term ϕ_c is obtained from the solution of transcendental equation $G(\phi_c) = 0$, where $G(\phi_c)$ is given by

$$G(\phi_c) = \epsilon\cos(\phi_{n+1}) - \epsilon\cos(\phi_n) -$$

$$-\frac{(\gamma-1)}{\delta(\gamma-2)}\Big[-V_n{}^{(2-\gamma)} + (V_n{}^{(1-\gamma)} + \delta\phi_c)^{\frac{(2-\gamma)}{(1-\gamma)}}\Big], \quad (2)$$

with $\phi_c \in (0, 2\pi]$.

(ii) Simple Collisions: the particle suffers a single collision with the moving wall and leaves the collision zone. In this case, $V_n{}^\star = V_A{}^{(1-\gamma)}$ and $\Delta T_n = \phi_T + \phi_c$, where

$$V_A = -\left[V_n{}^{(2-\gamma)} + \frac{\delta(\gamma-2)}{(\gamma-1)}(2-\epsilon-\epsilon\cos(\phi_n))\right]^{\frac{1}{(2-\gamma)}}$$

and

$$\phi_T = \frac{1}{\delta}\left\{\left[V_n{}^{(2-\gamma)} + \frac{\delta(\gamma-2)}{(\gamma-1)}2-\epsilon-\epsilon\cos(\phi_n)\right]^{\frac{(1-\gamma)}{(2-\gamma)}}\right\}-$$

$$-\frac{1}{\delta}\left\{V_n{}^{(1-\gamma)}\right\},$$

with ϕ_c obtained from solution of transcendental equation $F(\phi_c) = 0$, given by

$$F(\phi_c) = \epsilon\cos(\phi_{n+1}) - \epsilon -$$
$$-\frac{(\gamma-1)}{\delta(\gamma-2)}\left[-V_A{}^{(2-\gamma)} + (V_A{}^{(1-\gamma)} + \delta\phi_c)^{\frac{(2-\gamma)}{(1-\gamma)}}\right]. \quad (3)$$

A mapping for the simplified model is also constructed. We assume that both walls are fixed. One is at $x = l$ while the other is at the origin $x = 0$. However, when the particle collides with the wall located at the origin, it suffers a change of energy and momentum as if the wall was moving according to $x_w(t) = \epsilon\cos(\omega t)$, where ϵ is the oscillation amplitude and ω the frequency of oscillation. This approach retains the nonlinearity of the problem and avoids the inconvenience of finding numerical solutions for transcendental equations.

Considering dimensionless variables, the simplified mapping is given by

$$\mathrm{T}: \begin{cases} V_{n+1} = |-V_A - 2\epsilon\sin(\phi_{n+1})| \\ \phi_{n+1} = \phi_n + \phi_T \mod (2\pi) \end{cases}, \quad (4)$$

where

$$V_A = -\left(V_n{}^{(2-\gamma)} + \frac{2\delta(\gamma-2)}{(\gamma-1)}\right)^{\frac{1}{(2-\gamma)}}$$

and

$$\phi_T = \frac{1}{\delta}\left\{\left[V_n{}^{(2-\gamma)} + \frac{2\delta(\gamma-2)}{(\gamma-1)}\right]^{\frac{(1-\gamma)}{(2-\gamma)}} - V_n{}^{(1-\gamma)}\right\}.$$

In the simplified model, negative velocities are forbidden, because they are equivalent to a particle travelling outside the region between the walls. To avoid this problem, we arbitrarily insert the function module in the equation of velocity. The particle's velocity is reversed by the function module only if, after the collision, the particle is traveling in the negative direction. The function module has no effect on particle motion if it is moving in the positive direction after the collision.

The Jacobian matrix (J) for simplified mapping defined by the equations of V_{n+1}, ϕ_{n+1} is written as,

$$J = \begin{pmatrix} \frac{\partial\phi_{n+1}}{\partial\phi_n} & \frac{\partial\phi_{n+1}}{\partial V_n} \\ \frac{\partial V_{n+1}}{\partial\phi_n} & \frac{\partial V_{n+1}}{\partial V_n} \end{pmatrix}. \quad (5)$$

After some algebra, we find the determinant of the matrix (J), which is given by

$$det(J) = sign[-V_A - 2\epsilon\sin(\phi_{n+1})]\times$$
$$\times\left[-V_n{}^{(1-\gamma)}\left(V_n{}^{(2-\gamma)} + \frac{2\delta(\gamma-2)}{(\gamma-1)}\right)^{\frac{(\gamma-1)}{(2-\gamma)}}\right] \quad (6)$$

where the function $sign(u) = 1$ if $u > 0$ and $sign(u) = -1$ if $u < 0$. It is well known that the determinant of the Jacobian matrix for the conservative case is $det(J) = [V_n + \epsilon\sin(\phi_n)]/[V_{n+1} + \epsilon\sin(\phi_{n+1})]$. Comparing the determinant obtained from (4) with the determinant of the conservative case, we conclude that the shrinks area of the phase space. The term $\left[-V_n{}^{(1-\gamma)}\left(V_n{}^{(2-\gamma)} + \frac{2\delta(\gamma-2)}{(\gamma-1)}\right)^{\frac{(\gamma-1)}{(\gamma-2)}}\right]$ is the factor that prevents the area of phase space is preserved because it is $\neq 1$. We can see that in the limit of $\delta \to 0$, the results of conservative Fermi-Ulam model are recovered.

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