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PROPERTIES OF AN ARITHMETIC CODE FOR GEODESIC FLOWS

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1. INTRODUCTION

The application of differential geometry and topology concepts has led to new results in the physics of Hamiltonian dynamical systems. In particular, these concepts allow the identification of typical dynamical concepts with geometric and topological ones, *e.g.*, trajectories of a dynamical system are the geodesics in its phase space, when this space is equipped with a suitable metric. As a consequence, a geometric theory of chaotic dynamics emerges, becoming feasible a geometric interpretation of such systems [1].

The pioneering works on geodesic coding in hyperbolic spaces developed by Morse and Koëbe is still worth of the attention of mathematicians. In this direction, Series [2] and Adler and Flatto [3] propose a geodesic coding procedure based on the relationship identified by Bowen and Series [4] between the geodesic flow on a compact surface of constant negative curvature and the interval map defined by the Markovian partition of the Poincaré disc boundary; and relate their associated symbolic dynamics. The previous works explore the properties and interrelationships existing between these systems as for instance the use of ergodicity of the interval maps to prove ergodicity of geodesic flows and conversely; explicit formulas for invariant measures of interval maps from invariant measures for flows were also developed in these works.

Our aim is to apply the geodesic coding method proposed by Adler and Flatto in the analysis of dynamical systems taking into consideration their interpretation as geometric and topological structures. As a consequence, the characteristics of the geodesic coding and its capacity to convey information are established.

2. HYPERBOLIC FLOW AND GEODESIC CODING

The geodesic flow taken into consideration exists on a hyperbolic surface $\mathbf{S}$ of genus g. This surface has a planar description in the Poincaré disc $\mathbb{D}$ as a hyperbolic polygon (the fundamental region $\mathbf{F}$) whose edges are "glued" by a specific set of isometries $\mathbf{T}$ (considering the hyperbolic metric) in $\mathbb{D}$. $\mathbf{T}$ is finite and each of its elements glues an associated pair of edges of $\mathbf{F}$. This is illustrated in Figure 1, where the inner solid lines form the region $\mathbf{F}$ and T_1 is an element of $\mathbf{T}$ pairing the edges 1 and 7. The vertical and horizontal dashed straight lines show the edge-pairings of $\mathbf{F}$ by the isometries in $\mathbf{T}$. For the proposed coding procedure, polygons with $8g-4$ edges are considered. Figure 1 illustrates a polygonal region with 12 edges which is associated with a bi-torus, a surface of genus $g=2$.

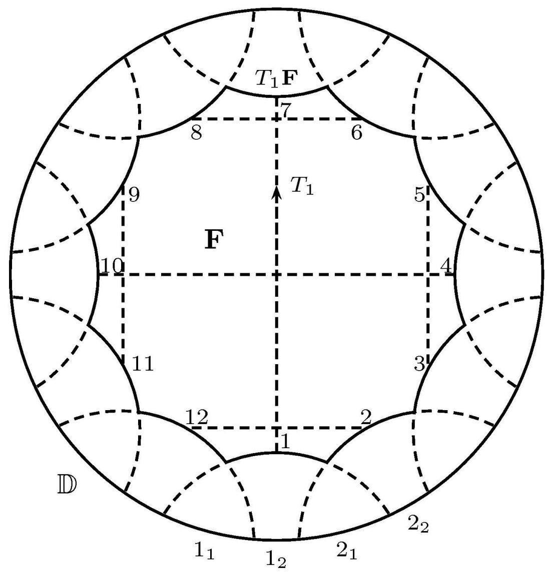

Figure 1 – Planar representation F of a surface S.

The analysis of the geodesic flow on $\mathbf{S}$ is done by considering a Poincaré map f defined by the elements of $\mathbf{T}$ ($f = T_i$ over the edge i of $\mathbf{F}$) and the Poincaré section is determined by the image of the edges of $\mathbf{F}$ over $\mathbf{S}$ by the gluing process.

A tessellation of $\mathbb{D}$, the union of non-overlapping images of $\mathbf{F}$ meeting only at vertex or edge, can be obtained by applying every possible isometry generated by finite concatenation of elements of $\mathbf{T}$ to the region $\mathbf{F}$. An example is shown in Figure 2, where part of $\mathbf{F}$ is shown, and adjacent to $\mathbf{F}$ the image of $\mathbf{F}$ by the application of an isometry T_i on $\mathbf{F}$, which

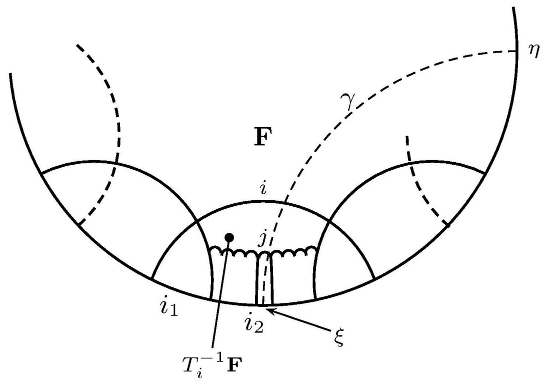

Figure 2 – Coding method.

is indicated with the label $T_i^{-1}\mathbf{F}$. The regions $\mathbf{F}$ and $T_i^{-1}\mathbf{F}$ have the same hyperbolic metric properties, however under a Euclidean perspective an exponential reduction of lengths and areas are observed. This fact is used to coding the extreme points ξ (forward) and η (backward) of a geodesic γ in $\mathbb{D}$, see Figure 2.

The coding process is similar to the binary expansion of the points in the unit interval $[0, 1)$ by the piecewise linear map $f(x) = (2x)$, where (α) denotes the fractional part of α. For this simple case, the subintervals $[0, \frac{1}{2})$ and $[\frac{1}{2}, 1)$ form a Markov partition of $[0, 1)$. Similarly, a Markov partition is induced in the boundary S^1 of $\mathbb{D}$ by extending (the dashed lines shown in Figure 1) the edges of $\mathbf{F}$ to S^1. The symbols i_1 and i_2, $1 \leq i \leq 8g-4$ (for the case $8g-4 = 12$) are the labels of the partition intervals. The interval map is given by the isometries in $\mathbf{T}$, where $f(x) = T_i(x)$ if x is in the edge i of $\mathbf{F}$ or the corresponding region of the Poincaré section of $\mathbf{S}$. Consider Figure 2. Let the region $\mathbf{F}$ be the reference after the n-th application of the transition map f, $n \in \mathbb{Z}$. Since the partition interval where the forward point ξ of the geodesic γ defined by ξ, η lies is i_2, it follows that if $w = \ldots w_{-1}w_0w_1w_2\ldots$ is the bi-infinite codeword of γ, then $w_n = i_2$. For the next step, the region of the tesselation of $\mathbb{D}$ defining the partition of the interval i_2 of S^1 is $T_i^{-1}\mathbf{F}$ due to the fact that the geodesic enters this region after leaving $\mathbf{F}$ by crossing the edge i. Now, either ξ belongs to the interval j_1 or to the interval j_2 of the partition defined by $T_i^{-1}\mathbf{F}$ in the previous defined region i_2, thus w_{n+1} is equal to j_1 or j_2, depending on the region where ξ belongs. The same idea follows for the expansion of both extremes of the geodesics ξ and η, and the concatenation of these expansions determines the bi-infinite code sequence of the geodesic.

The space of sequences coding the geodesics forms a shift space X (see [3]). In fact, X is a Markov shift with possible transitions given by (1).

$$\mathcal{T}: \begin{cases} i_1 \to (\sigma(i)+1)_2,\ (\sigma(i)+2)_1. \\ i_2 \to (\sigma(i)+2)_2,\ (\sigma(i)+3)_1, \ldots, \\ \qquad (\sigma(i)-2)_1,\ (\sigma(i)-2)_2. \end{cases} \tag{1}$$

where

$$\sigma(i) = \begin{cases} 4g - i \mod (8g-4), & i \text{ odd}, \\ 2 - i \mod (8g-4), & i \text{ even}. \end{cases}$$

3. RESULTS

Knowing the properties of a set of code sequences is a first step in the analysis of the characteristics of a code. With the purpose of applying the geometric and algebraic properties of hyperbolic manifolds to analyze dynamical systems, through the symbolic sequences coding the geodesics of a hyperbolic manyfold, we determine two characteristics of the set of symbolic sequences. The first characteristic is associated with a general aspect of the set of sequences, and specify (as a function of the genus of $\mathbf{S}$) the maximum full-shift embedded in the shift space generated by the geodesic coding process. This result is established in Theorem 1.

Theorem 1. *The cardinality $|\Sigma|$ of the alphabet of the maximum* full-shift *embedded in the shift space X is equal to* $4(g-1)$.

The second characteristic considers a bound on the coding design, that is, the maximum capacity to convey information through the set of code sequences. This result is stated in Theorem 2. This capacity is known as the topological entropy of the shift space (see [5]).

Theorem 2. *The topological entropy of the symbolic dynamical system X as a function of the genus g of the surface associated with a regular fundamental region $\mathbf{F}$ with $(8g-4)$ edges, is given by*

$$h(X) = \log\left[(4g-3) + \sqrt{(4g-3)^2 - 1}\right].$$

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