RELATIVISTIC EFFECTS, LENGTH SCALE, QUASIPERIODICITY AND CHAOS IN THE ZAKHAROV EQUATIONS

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Abstract

Nonlinear saturation of unstable solutions to the full, one-dimensional, weakly relativistic Zakharov equations is considered in this paper. Focusing attention on instabilities developing from low-frequency perturbations we introduce a quantity proportional to the initial energy of the high-frequency field, ρ_{oo}, *and the wave vector of the basic perturbing low-frequency mode,* $k = 2\pi/L$ *with* L *as the length scale, to identify a number of regions on a* $\rho_{oo} \times k$ *parametric plane. For* $\rho_{oo} \ll 1$ *ion-acoustic dynamics is found to be unimportant no matter the value of* k*; in this situation the system is numerically shown to be integrable even for very small values of* k *where the solutions are not simply periodic. For larger values of* ρ_{oo} *ion-acoustic fluctuations become active modes of the system driving a transition to chaos if* k *is bellow a critical value. The influence of relativistic terms in the Zakharov equation is investigated; it is shown that such terms generally lower the amplitude thresholds for transition to chaos.*

Langmuir turbulence has been one of the most studied problems in modern nonlinear plasma physics. Over the last years a great deal of effort has been directed to the analysis of the turbulence and related subjects as cavitation, collapse, soliton dynamics and others [1, 2]. The turbulence is governed by the Zakharov equations which nonlinearly couples a high-frequency electrostatic field with frequencies close to the plasma frequency (the Langmuir field), to low-frequency fluctuations with time scales comparable to the ion-acoustic period (the ion-acoustic field) [3]. When the time scale of the ion-acoustic field is sufficiently slow, a condition characterizing the so called sub-sonic regime, the governing set of equations reduces to a single Nonlinear Schrödinger Equation (NLS). In this case the low-frequency modes are enslaved to the high-frequency ones and the system is shown to be completely integrable. On the other hand, when the ion-acoustic time scale is not slow enough, low-frequency modes become active, destroying integrability and driving a transition to chaos. An useful integrable low-dimensional model can be obtained as follows. One feeds a certain amount of energy into a stationary high-frequency Fourier mode and perturb this equilibrium solution with a low-frequency disturbance consisting of a single wave vector mode k which sets the length scale of the problem. Starting from this initial situation, and imposing some restrictions on the initial amplitude of the high-frequency field and on the wave vector of the low-frequency mode, the Fourier series of the nonlinear developing solutions are then truncated to three modes. Two are the restrictions justifying the truncation procedure: (i) the possibility of enslaving the low-frequency dynamics to the high-frequency one, whereupon by disregarding time derivatives in the equation for density fluctuations one obtains the NLS mentioned above; and (ii) choosing the perturbing wave vector such that its harmonics are outside the unstable band. In the present paper it is found that both restrictions can be easily represented on a parametric bidimensional space where one of the axis is proportional to the initial energy and the other is the wave vector k. It is shown that the low-dimensional domain is a finite region of this parametric space. As one abandons the low-dimensional region, solutions are likely to loose their regularity. It shall be seen that this is indeed the fact. With numerical simulations and the analysis of Lyapunov diverging trajectories we shall see that the appearing irregular solutions can be of two types: (i) For very small values of the initial energy, the irregular solutions are quasiperiodic with no associated chaos; (ii) for larger values, solutions are chaotic with positive Lyapunov exponents. Relativistic terms are shown to lower the amplitude thresholds to chaos. As they can thus alter the character of an outcoming solution from regular to chaotic, they are to be considered as relevant for the problem. Note that even if the original perturbing terms are small, the final effect cannot be simply allocated in simple small corrections of phases or amplitudes of the solutions. Let us first write down the conveniently normalized form of the weakly relativistic Zakharov equation which shall be used in the following analysis. These equations are [4]:

$$i\partial_t E + \partial_x^2 E = n\ E - \alpha |E|^2 E \tag{1}$$

and

$$\partial_t^2 n - \partial_x^2 n = \partial_x^2 |E|^2, \tag{2}$$

where E is the electric field normalized to $8\sqrt{\pi \frac{1}{3}\frac{m_e}{m_i} n_o T_e}$ with $m_{e,i}$ as the electron/ion mass and T_e as the electron temperature. n is the low-frequency density fluctuations normalized to the equilibrium density n_o and α is the weakly relativistic parameter given by $\alpha \equiv 3T_e/m_e c^2 \ll 1$ with c the velocity of light. Time and coordinate are adimensionalized as $(m_e/m_i)\omega_p t \rightarrow t$ and $\sqrt{m_e/m_i}(\omega_p/v_e)x \rightarrow x$ respectively, where ω_p is the electron plasma frequency. It is now assumed that the solution is spatially periodic with period $L = 2\pi/k$ where $2\pi/k$ is the basic length scale. At this point the low-dimensional model can be obtained if one Fourier expands all the dynamical variables and truncates the series to three high-frequency terms, $E = E_o(t) + E_+(t)e^{ikx} + E_-(t)e^{-ikx}$. Besides, one assumes that the time scale satisfies $\partial_t^2 << \partial_x^2$, to get $n = n_1 e^{ikx} + n_2 e^{2ikx} + c.c.$ with

$$n_1 = -E_+ E_o^* - E_-^* E_o, \qquad n_2 = -E_+ E_-^*.$$

In view of the fact that for a real variable $n_p = n_{-p}^*$, these two relations are sufficient to determine the low-frequency field. On substituting n_1 and n_2 into Eq.(1), writing

$$E_j(t) \equiv \sqrt{\rho_j}(t) e^{i\phi_j(t)} e^{\int^t \alpha(|E_o|^2 + |E_+|^2 + |E_-|^2)dt},$$

whith $j = 0, +, -$, defining $\psi \equiv \phi_+ + \phi_- - 2\phi_o$ and separating the low-dimensional group of equations into real and imaginary parts, one arrives at the following equation for the variable ρ_o solely: $d_t\rho_o = -2(1+\alpha)\rho_o(\rho_{oo} - \rho_o)\sqrt{1-\cos^2\psi}$, where $\cos^2\psi = ((\frac{k}{1+\alpha})^2 - \rho_o - (\frac{\rho_{oo}-\rho_o}{4}))/2$. If one introduces $\rho_{oo} \equiv \rho_o(t=0)$, the validity conditions for the various formulas obtained so far generate the set of relations:

$$k < k_s(\rho_{oo}) \equiv \sqrt{2(1+\alpha)\rho_{oo}}, \qquad k > k_h(\rho_{oo}) \equiv \sqrt{\frac{(1+\alpha)\rho_{oo}}{2}}, \qquad k \gg k_i(\rho_{oo}) \equiv 2\sqrt{1+\alpha}\rho_{oo}.$$

We show the region defined by curves k_s, k_h and k_i above in Figure 1, both for the nonrelativistic $\alpha = 0$ case and for a weakly relativistic case $\alpha = 0.3$. From k_i one sees that the low-dimensional region is in fact situated at $\rho_{oo} \ll 1$. When such a condition is satisfied, the system is expected to be stable above k_s, simply periodic between k_s and k_h and quasiperiodic bellow k_h. Curve k_i defines the line of transition between subsonic and supersonic ion-acoustic fluctuations. Ion acoustic modes become active if one is close to k_i and begins to play some noticeable role in the dynamics for $\rho_{oo} \sim 1$ where k_i becomes comparable to the k_s or k_h. As mentioned before, relativistic effects are expected to lower the thresholds for transition to chaos. Let us proceed to check all the assertives above with

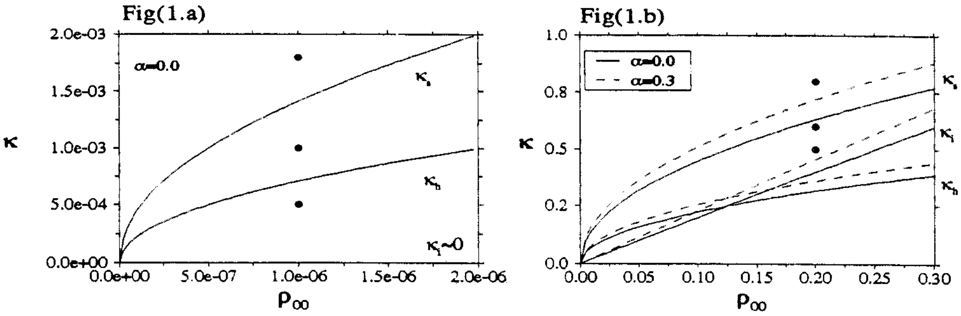

Figure 1: Parametric phase-space $\rho_{oo} \times k$ for $\rho_{oo} \ll 1$ (fig. (a)) and $\rho_{oo} \sim 1$ (fig (b)). The filled circles show the simulated points.

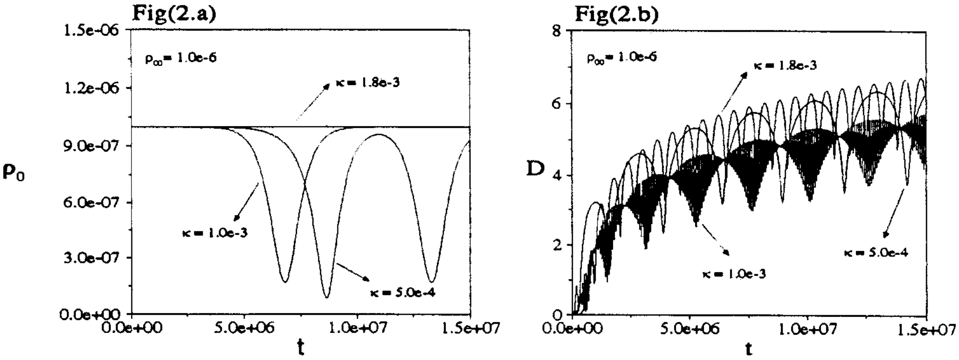

Figure 2: Time Series of $\rho_o(t)$ (a) and $D(t)$ (b) for $\rho_{oo} = 1.0e{-6}$, $k = 1.8e{-3}, 1.0e{-3}, 5.0e{-4}$.

numerical simulations based on the full set (1) and (2). Our simulation scheme consists in writing all the dynamical variables and the differential equations as Fourier series in the spatial variable. A number of modes ranging from 32 to 128 for each dynamical variable is used, nonlinear products in the differential equations are evaluated with a FFT subroutine and the set of temporal equations is advanced in time with a predictor-corrector algorithm. Both the FFT and the predictor-corrector are subroutines of a CRAY YMP-2E computer. Numerical precision is tested by monitoring the time evolution of the conserved quantity $\mathcal{H} = \int_o^L[|\partial_x E|^2 - \alpha|E|^4 + n|E|^2 + \frac{1}{2}(n^2 + v^2)]dx$, with $\partial_t n = -\partial_x v$; relative errors were found to be about one part in 10^8. As a first investigation, let us examine the behavior of the system for small values of ρ_{oo}, $\rho_{oo} = 1.0e - 6$. We do so in Figure (2a) for $k = 1.8e - 3$, $k = 1.0e - 3$ and $k = 5.0e - 4$. In the first case whose representative point lies in the stable region of Figure 1, above k_s, one has indeed a trivially stable solution. The second case corresponds to a point between k_s and k_h - in agreement with the estimates, one has a periodic solution. In the third case one is bellow k_s and the solution appears to be nonperiodic or irregular. We now introduce the Lyapunov coefficient as the average slope of the function $D(t) = log[d(t)/d(t = 0)]$ in a $D(t) \times t$ plot. $D(t)$ is the Euclidean distance between two initially close trajectories, calculated according ref. [1]. In Figure (2b) we plot $D(t)$ corresponding to the three cases of Figure (2a). The plots show that in all cases the diverging trajectories goes at most as $log(t)$, which signals the existence of integrable dynamics [5]. The irregular integrable trajectory occurring for $k = 5.0e - 4$ is therefore a quasiperiodic one, a feature to be expected in view of the properties of the NLS governing the system for such a low values of ρ_{oo}. We repeat the simulations in Figure 3 now for $\rho_{oo} = 0.2$. Again, for $k > k_s$ one has a stable solution. However, the solution corresponding to $k_h < k = 0.5 < k_s$ is chaotic. This can be clearly seen from Figure (3b) where a positive

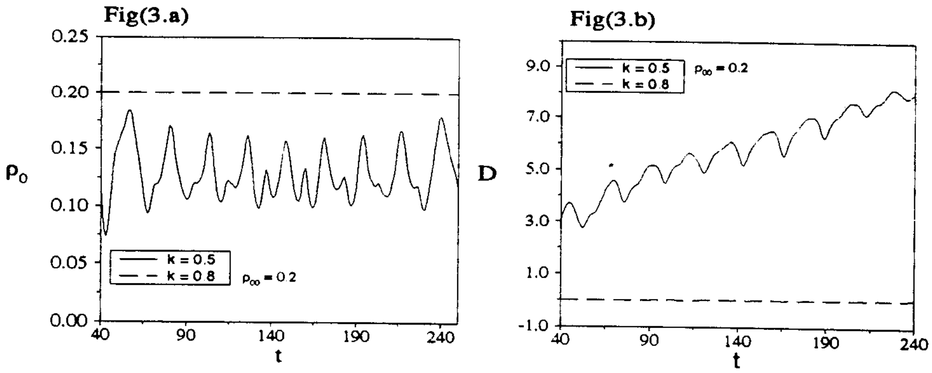

Figure 3: Same as in Figure 2 for $\rho_{oo} = 0.2$, $k = 0.8, 0.5$.

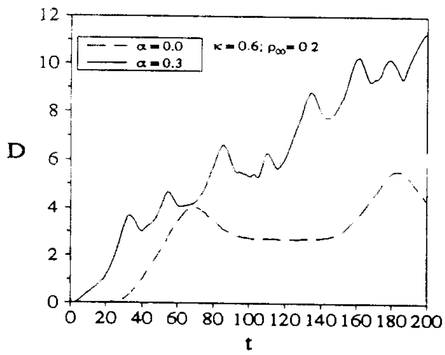

Figure 4: De-stabilizing influence of relativistic effects: $\alpha = 0$ (dashed line); $\alpha = 0.3$ (full line).

average slope of $D(t) \times t$ can be identified. Further reduction of k bellow k_h causes no additional alteration on the character of the chaotic solutions. A final simulation represented in Figure 4 is performed for $k = 0.6$ and $\rho_{oo} = 0.2$ where the nonrelativistic $\alpha = 0$ dynamics is regular. In agreement with the estimates, the Figure shows that if α is raised to $\alpha = 0.3$, chaos is brought in. More detailed investigation on the problem shall be published elsewhere.

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