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EQUILIBRIUM POINTS AROUND A DOUBLE ROTATING MASS DIPOLE

Santos, L. B. T.[1]
Prado, A. F. B. A.[1]
Sanchez, D. M.[1]
[1]National Institute for Space Research, INPE, São José dos Campos, SP, Brazil
leonardo.btorres@inpe.br.
antonio.prado@inpe.br.
diogo.sanchez@inpe.br.

Abstract. *The objective of the present work is to determine the equilibrium points and to perform an analysis of zero velocity curves in the Modified Restricted Synchronous Three-Body Problem. To perform this task, it is necessary to obtain the equations of motion, in a rotating reference frame, of a spacecraft with negligible mass that orbit around a system constituted of two massive bodies. The two massive bodies are assumed to have irregular shaped bodies and were modeled as a rotating mass dipole. The exact positions of the equilibrium points are obtained and also the values of the Jacobi constant necessary to emerge each equilibrium point. We analyzed how the position of the collinear equilibrium points vary when we modify the size and mass ratio of the two primary bodies.*

Keywords: *Equilibrium Points, Zero Velocity Curves, Modified Restricted Synchronous Three-Body Problem*

1. INTRODUCTION

It is notorious how much has grown the scientific interest in the exploration of asteroids in the last years (Pamela and Misra, 2011). Asteroid exploration has brought information about the dynamics and formation of asteroids and has entailed to a better understanding of the origin of the solar system (Pamela *et al.*, 2013).

The Exploration of asteroids and comets is a task quite challenging, due the fact of each body has its own physical characteristics of shape, density, rotation, mass distribuition, et al. Then, develop a mathematical modeling of gravitation field close these bodies is a task quite complex and is necessary develop this equations encompassing the maximum of parameters possible, because several times the physical characteristics of asteroids are only discovered after an approaching spacecraft (Scheeres *et al.*, 2000).

The model adopted here assumes that the two asteroids of the binary system are modeled as a rotating mass dipole, with the purpose of representing a natural elongated body (Zeng *et al.*, 2016a). Initially, this mathematical approach in an asteroid was introduced by Zeng *et al.* (2015). More current studies of Zeng *et al.* (2016b) have adopted an improved dipole model, in which they assumed a flat dipole in an asteroid. In this study, Zeng *et al.* (2016b) found up to 13 libration points in the plane of motion. Ferrari *et al.* (2016) investigated a way to find models in the trajectories next to an asteroid system using the dipole model of rotating mass. Santos *et al.* (2017) investigated the equilibrium points and their respective zero velocity curves of an asteroid binary system, considering the restricted three-body synchronous problem, in which one of the asteroids was considered as a point mass and the second asteroid was considered as a mass dipole in rotation. In this work, we investigated the equations of motion in an asteroid binary system which the two asteroid are modeled as dipole mass in rotation.

2. EQUATIONS OF MOTION

The Modified Restricted Synchronous Three-Body Problem has the objective of describing the dynamics of an infinitesimal mass particle (P) that moves under the gravitational influence of two other massive bodies (M_1 and M_2) that orbit around the center of mass of the system. The distance unit is normalized by the distance from the center of mass of the body M_1 to the center of mass of the body M_2. The two primary bodies are modeled as a rotating mass dipole, i. e., each primary body is formed by two hypothetical bodies with masses m_{11} and m_{12} (for body M_1) and m_{21} and m_{22} (for body M_2), as shown in Figure 1.

The mass ratio is given by

$$\mu^* = \frac{m_{21}}{m_{11} + m_{12} + m_{21} + m_{22}}. \tag{1}$$

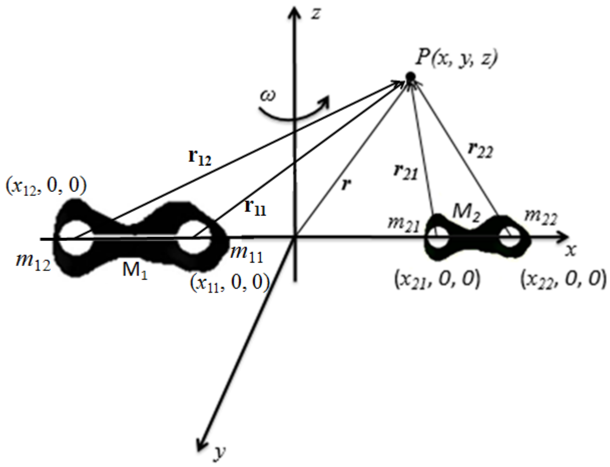


Figure 1: Representative image of the geometric shape of the system under study (not in scale).

The primary bodies are located on the x-axis, whose coordinates are given by

$$x_{11} = -2\mu^* - d_1/2, \tag{2}$$

$$x_{12} = -2\mu^* + d_1/2, \tag{3}$$

$$x_{21} = -2\mu^* - d_2/2 + 1, \tag{4}$$

$$x_{22} = -2\mu^* + d_2/2 + 1. \tag{5}$$

Here d_1 is the distance between the point of mass m_{11} and m_{12} (dimension of M_1) and d_2 is the distance from the point of mass m_{21} to the point of mass m_{22} (dimension of M_2). The equations of motion of the body of negligible mass, when viewed from a rotating reference, are given by

$$\ddot{x} - 2\dot{y} = \Omega_x, \tag{6}$$

$$\ddot{y} + 2\dot{x} = \Omega_y, \tag{7}$$

where

$$\Omega = \frac{x^2 + y^2}{2} + \frac{1 - 2\mu^*}{2r_{11}} + \frac{1 - 2\mu^*}{2r_{12}} + \frac{\mu^*}{r_{21}} + \frac{\mu^*}{r_{22}}, \tag{8}$$

with

$$r_{11} = [(x - x_{11}),\ y,\ 0]^T, \tag{9}$$

$$r_{12} = [(x - x_{12}),\ y,\ 0]^T, \tag{10}$$

$$r_{21} = [(x - x_{21}),\ y,\ 0]^T, \tag{11}$$

$$r_{22} = [(x - x_{22}),\ y,\ 0]^T, \tag{12}$$

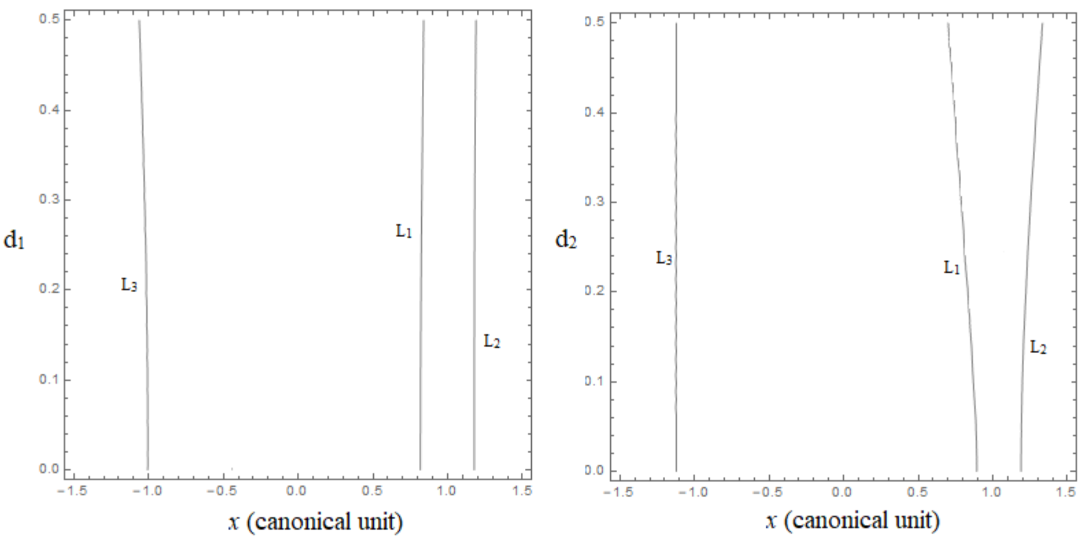
in that Ω_x and Ω_y are the partial derivatives of Ω with respect to x and y, respectively.

The known *Jacobi's Integral* (Szebehely, 1967) is given by:

$$v^2 = 2\Omega - C. \tag{13}$$

Note that the Equation 13 is dependent on the function Ω and an integration constant *C*, which is an integral of the equations of motion. In the literature, *C* is known as *Jacobi's Constant* (Dutt and Anilkumar, 2014; Ren and Shan, 2014). In this work we are assuming that the spacecraft is moving in the plane *xy*. Then, the Equation 13 shows us that the spacecraft velocity is a function of the position of the body in the plane, for a given numerical value of *C* (generated from the initial conditions) (McCuskey, 1963). It is observed that in Equation 13 there is a relation between the square of the velocity and the positions of the spacecraft in the rotating coordinate system (Molton, 1960).

In mathematical terms, the $zero\ velocity\ curve$ (ZVC) is defined as $2\Omega - C = 0$ (McCuskey, 1963; Szebehely, 1967). The ZVC expressed in Cartesian coordinates is given by



(a) Variation of collinear equilibrium points as a function of the size of the most massive primary.

(b) Variation of the collinear equilibrium points as a function of the size of the less massive primary.

Figure 2: Variation of the collinear equilibrium points taking in consideration the dimension of the primary bodies.

$$\frac{x^2+y^2}{1}+\frac{(1-2\mu^*)}{r_{11}}+\frac{(1-2\mu^*)}{r_{12}}+\frac{2\mu^*}{r_{21}}+\frac{2\mu^*}{r_{22}}=C. \tag{14}$$

The regions in which the spacecraft movement is allowed are regions where $2U > C$, otherwise, by Equation 13, we see that the square of the velocity would become negative, which is a physical impossibility (Szebehely, 1967).

Note that it is not possible to obtain, through this analysis, any information about the specific trajectory of the particle studied. Only the limits are determined (Szebehely, 1967). The contour curve of the equation 14 shows us the border regions where motion is allowed (Dutt and Sharma, 2011; McCuskey, 1963).

3. RESULTS AND DISCUSSION

Figure 2a and 2b show the x coordinates of points L_1, L_2 and L_3 for different values of d_1 and d_2, respectively. Note that in Figure 2a (where we vary the dimension of M_1) the effects slightly modify the positions of L_3 and L_1. L_2 is far from M_1, so it remains practically constant. In Figure 2b the size of M_2 was modified. Due to this fact, the points L_1 and L_2 have larger influence (because they are closer to the M_2) due to the new mass distribution of this body. Notice that L_3 remains practically constant.

In Figure 3, it shows how the position of the collinear points vary when we increase the mass ratio of the system.

We can note from Figure 3 that as we grow the mass ratio μ^*, the mass of the M_2 increase and the mass of M_1 decrease. With this change of gravitational field of both the bodies, this causes a new configuration to be required to establish the equilibrium points. Notice that as we increase the mass of M_2, the equilibrium points that are closest to this body move away, causing the equilibrium point L_1 to move to the left and the equilibrium point L_2 to move to the right with respect to M_2. The equilibrium point L_3 is more distant from the primary, which makes it difficult to understand the behavior of this body intuitively. Numerical evidence shows that as we increase the mass ratio, the point of equilibrium L_3 moves away from the primary bodies, causing a new relation between centrifул force and gravitational force to be necessary to annul generating a stationary point.

Some numerical tests were performed, where we consider $\mu^* = 0.0049505$, $d_1 = 0.736068$ and $d_2 = 0.131440$. These numerical values are based on the $Alpha - Gamma$ asteroid pair of the $2001SN_{263}$ asteroid system. Doing the right side of Equations 6 and 7 equal the zero and solving numerically, we find five real roots, which three of them are collinear (L_1, L_2 and L_3) and the another two are in plane xy. The localization these equilibrium points are shown in Table 1. In Figure 4 show the positions of the equilibrium points (red) with respect to the center of mass of each primary body (black).

Some ZVC are shown in next. The color coded indicates the velocity that the spacecraft will have in each region. The color column legend shows the square velocity needed to cross from one region to another. The red regions are forbidden regions where the moviment is not possible. For a spacecraft reach the in red regions with the initial conditions generated, it is necessary has velocity square negative, which is a physical impossibility.

In Figure 5 we can observe that the ZVC related the energy C_1 = 3.716359670795 touches on a point called L_1. The

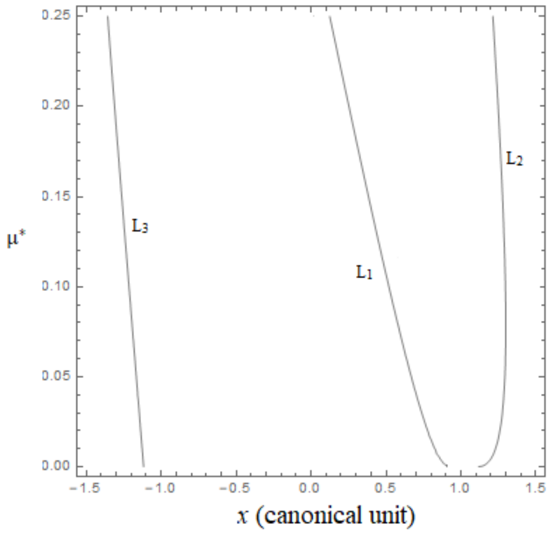


Figure 3: Positions μ^* of collinear equilibrium points (L_1, L_2 and L_3) for different values of μ^*.

Table 1: The positions of equilibrium points for the studied system

		Equilibrium points
L_1	x	0.8621142586696
	y	0
L_2	x	1.2000933511901
	y	0
L_3	x	-1.122101868767
	y	0
L_4	x	0.0046508345280
	y	0.9276535170573
L_5	x	0.0046508345280
	y	-0.9276535170573

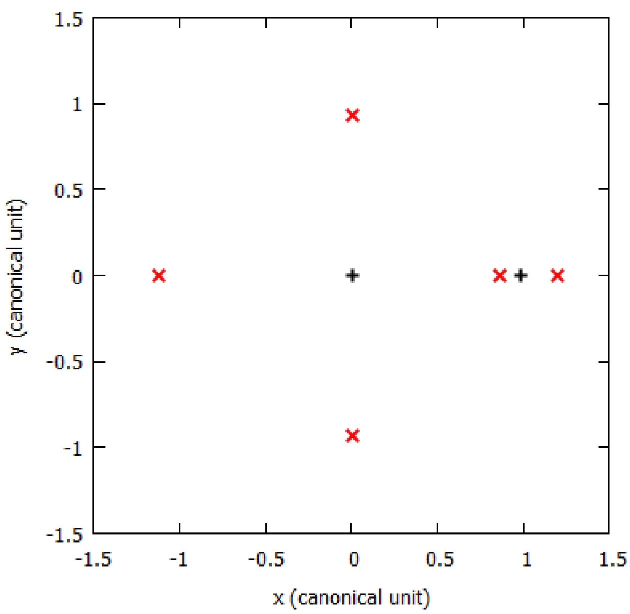
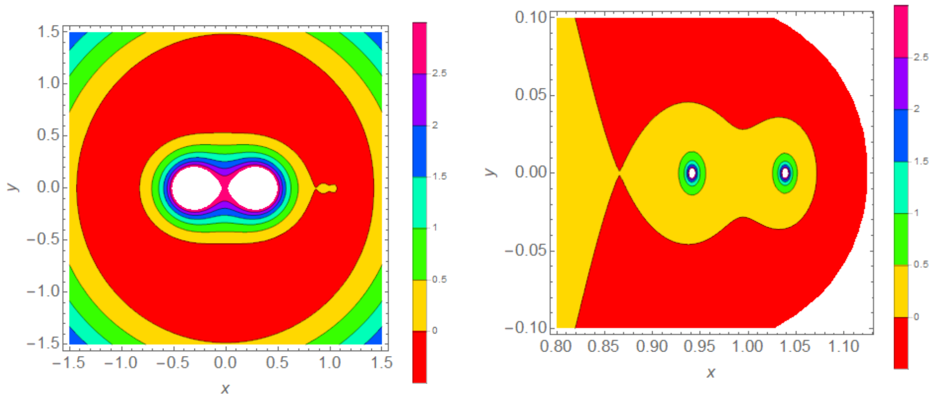


Figure 4: Positions of the equilibrium points for the system studied (red) and the positions of center of mass of primaries bodies (green).

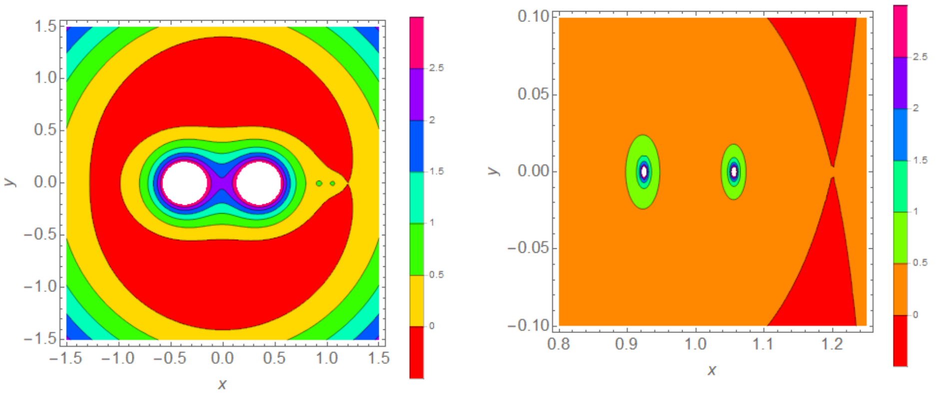
value C_1 become possible the transfer of spacecraft between M_1 and M_2 through of L_1 point. The Figure 5b show a visual aproximation of figure 5a in M_2 for a better visualization of ZVC close of the M_2.



(a) *Zero velocity curves*.

(b) Magnification of Figure 5a, showing the region around M_2.

Figure 5: *Zero velocity curves*. The first point of contact occurs in C = 3.716359670795.



(a) *Zero velocity curves*.

(b) Magnification of Figure 6a, showing the region around M_2.

Figure 6: *Zero velocity curves*. The second point of contact occurs in C = 3.34813335305.

It is important observe that the spacecraft coming from M_1 initially and with energy C_1, for example, just can reach the body M_2 only crossing by L_1 point. We can note that, by figure Plau that the transfer between the regions close M_1-M_2 and the infinite remains forbidden, but the transfer between M_1 and M_2 is possible.

Decreasing the value of C, the curves close to M_1 and M_2 (oval inner) becomes greater and the external curve become smaller. When the value of C arrive in C_2 = 3.34813335305, the inner oval and external oval get in touch so, we have a second contact point. This contact point is called L_2. It is possible note that for $C = C_2$ become possible a comunication between the regions close of $M_1 - M_2$ with the infinite. The Figure 6b is a visual aproximation of Figure 6a around of body M_2.

Decreasing even more the value of C util C_3 = 3.2678562132, the prohibited region reduce (red region smaller than in Figures 5 and 6) and the spacecraft has more regions for moviment, as shown the Figure 7. We can note that there is a link between the region close to M_1 and the infinite, but this time from left side. This contact point is called L3.

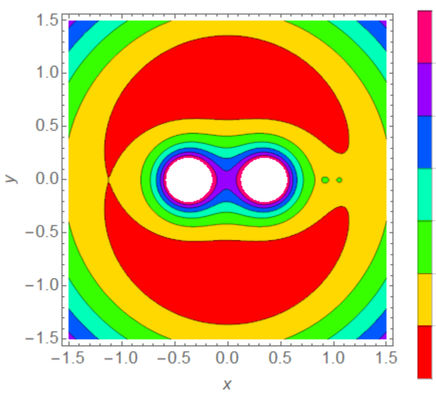


Figure 7: *Zero velocity curves*. The third point of contact occurs in C = 3.2678562132.

Finaly, decreasing of the C_3 to $C_{4-5} = 2.85925963595$, the forbidden region becomes smaller when compared to previous cases. Note that, only the regions around of the L_4 and L_5 points remain as prohibit, as shown in Figure 8. The notation C_{45} is a form abbreviated of write C_4 and C_5 because they have the same values.

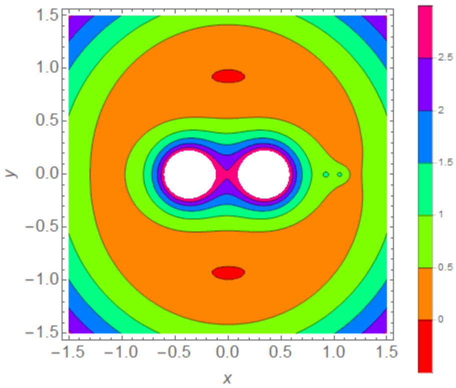


Figure 8: *Zero velocity curves*. The four and five points of contact occurs in C = 2.85925963595.

4. CONCLUSIONS

In this work we investigate the influence that the dimension of the primary bodies and their respective masses has on the generation of the equilibrium points. For the quantitative study we realized that as we increase the dimension of M_2, the equilibrium points L_1 and L_2 are more influenced (because they are closer to this body) and move away from M_2. On the other hand, the point of equilibrium L_3 does not undergo a considerable change by being more distant. On the other hand, when we vary the dimension of the M_1, we note a considerable change in position of equilibrium point L3. However, the positions of equilibrium points L_1 and L_2 practically not alter.

It was also analyzed the influence that the mass ratio μ^* affects the positions of the equilibrium points. Numerical evidence shows that as we increase the mass ratio (the mass of M_2 becomes larger), all collinear equilibrium points move away from M_2. This is due to the fact that the gravitational field of M_2 becomes greater (with the increase of μ^*) and with that, a new configuration is necessary to establish the points of equilibrium.

The zeros velocity curves were obitained in order to find the regions in which the spacecraft movement in the xy plane is allowed for different values of C. In this study, we verified that as we decrease the value of the Jacobi constant C, the regions where movement is permitted becomes greater.

5. ACKNOWLEDGMENTS

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